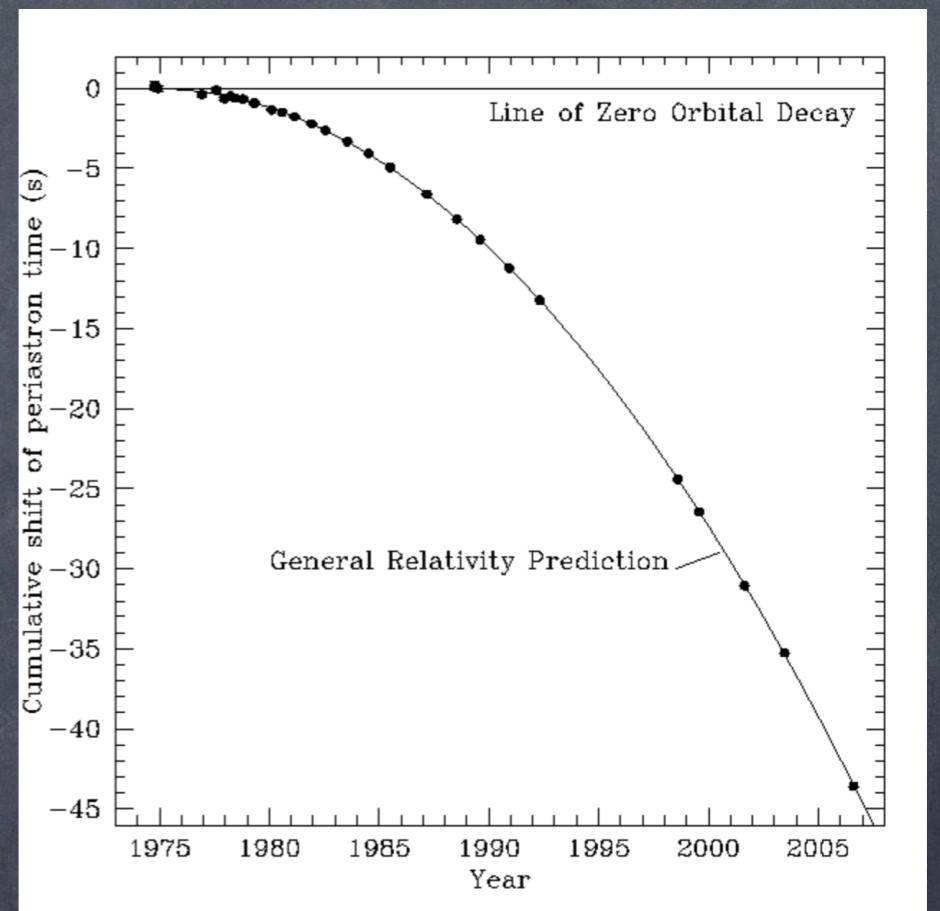


# Lectures on Gravitational Waves

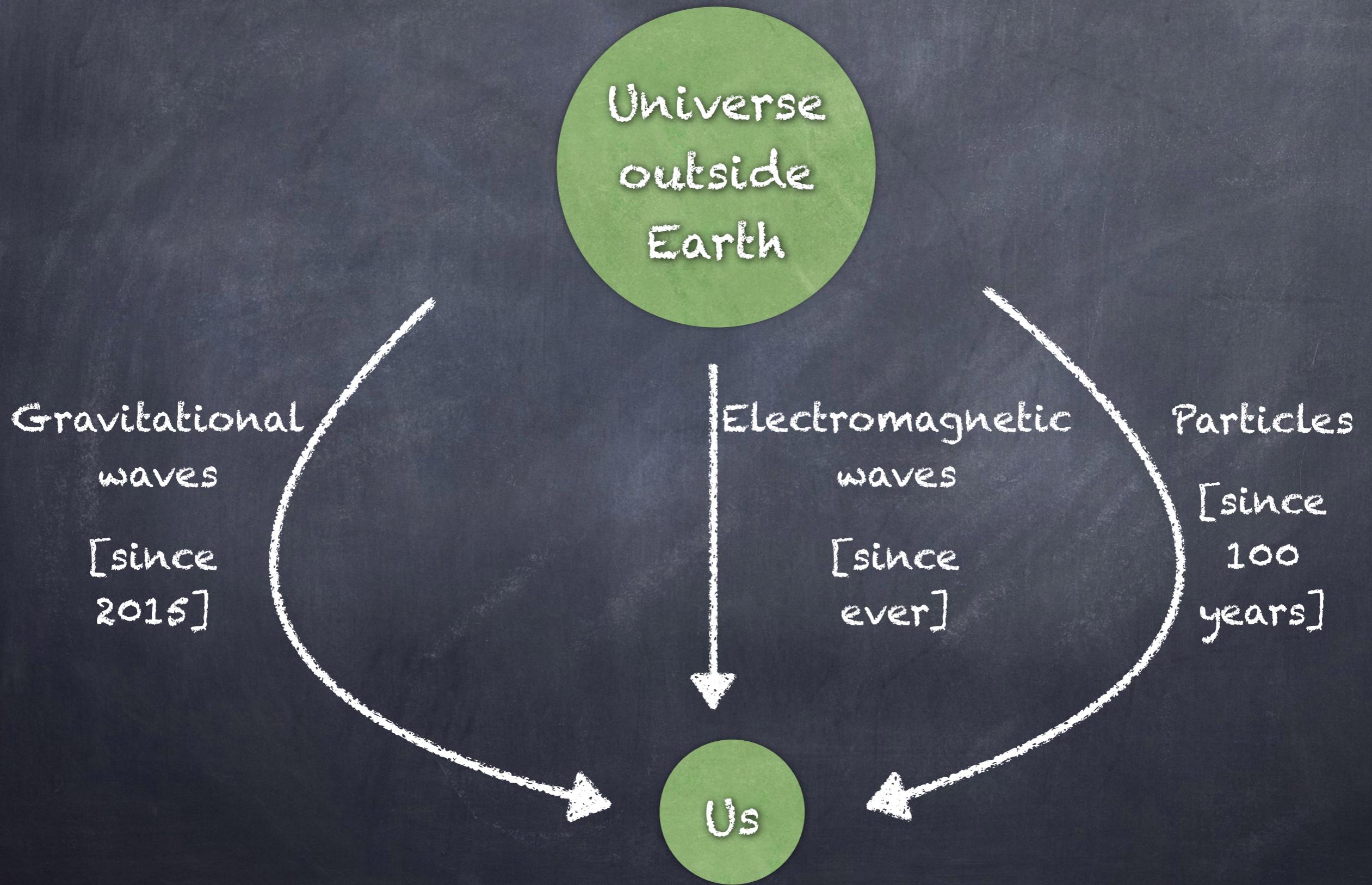
Geoffrey Compère  
Solvay Doctoral School  
October, 2022

# Exciting times...

- 1915 Birth of General Relativity and in 1916: prediction of Gravitational Waves
- 1974 Hulse-Taylor pulsar : First indirect detection of GW
- 2015 First direct detection of binary black hole merger: event GW150914
- 2017 First direct multi-messenger detection of binary neutron star merger: event GW170817
- March 2023 Run 04 of adLigo-adVirgo-Kagra

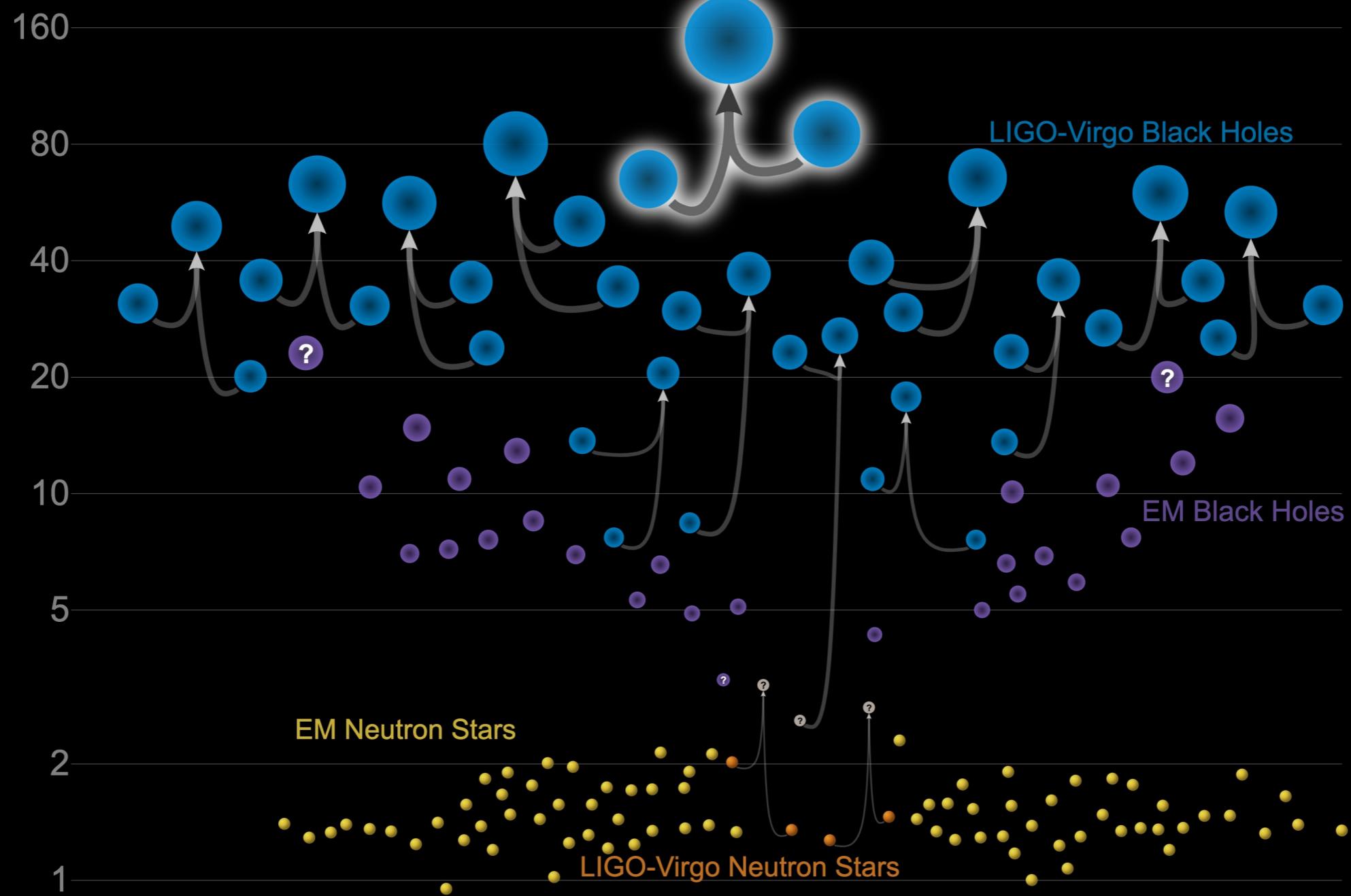


# The three messengers



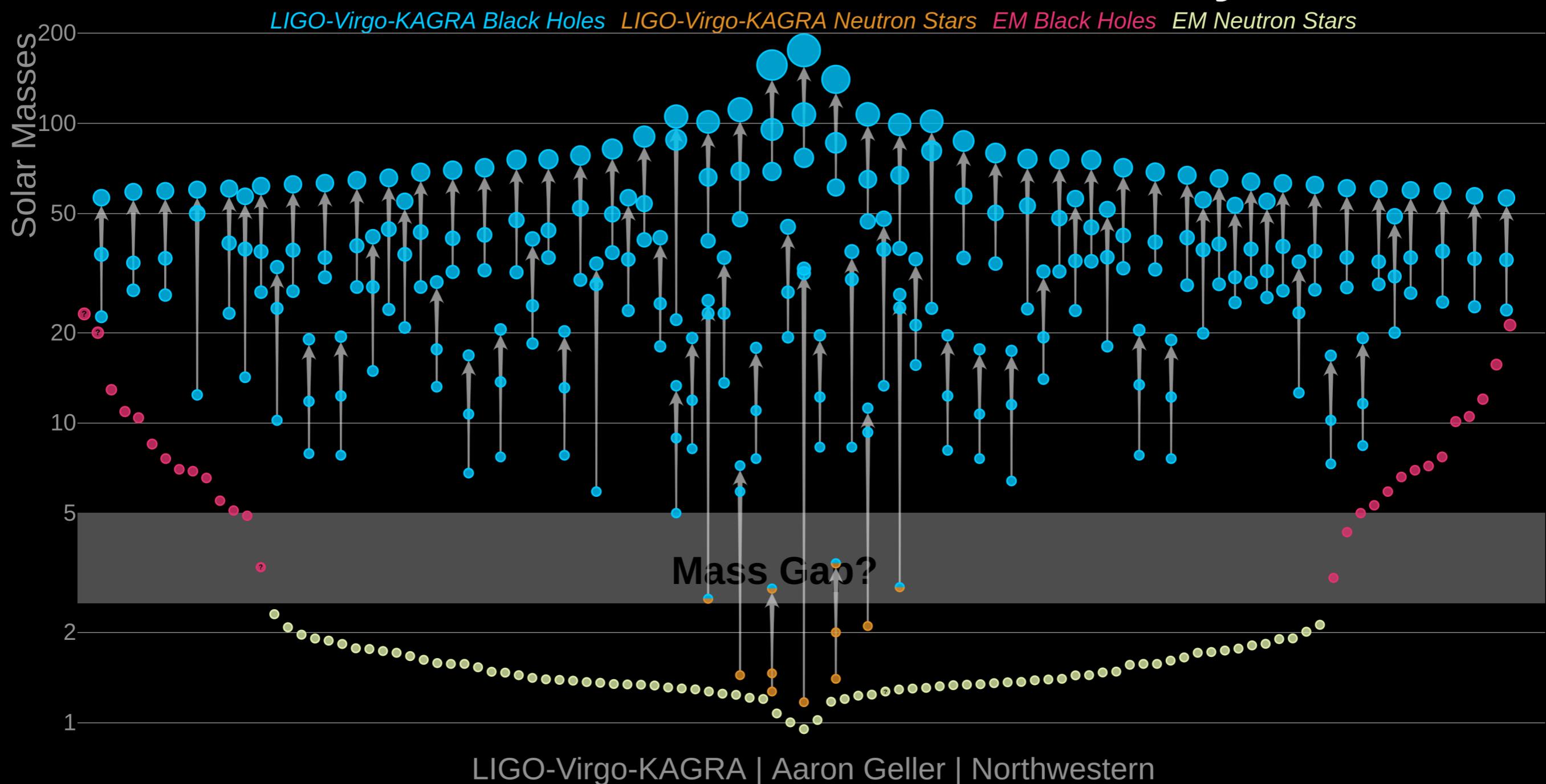
# Masses in the Stellar Graveyard

*in Solar Masses*



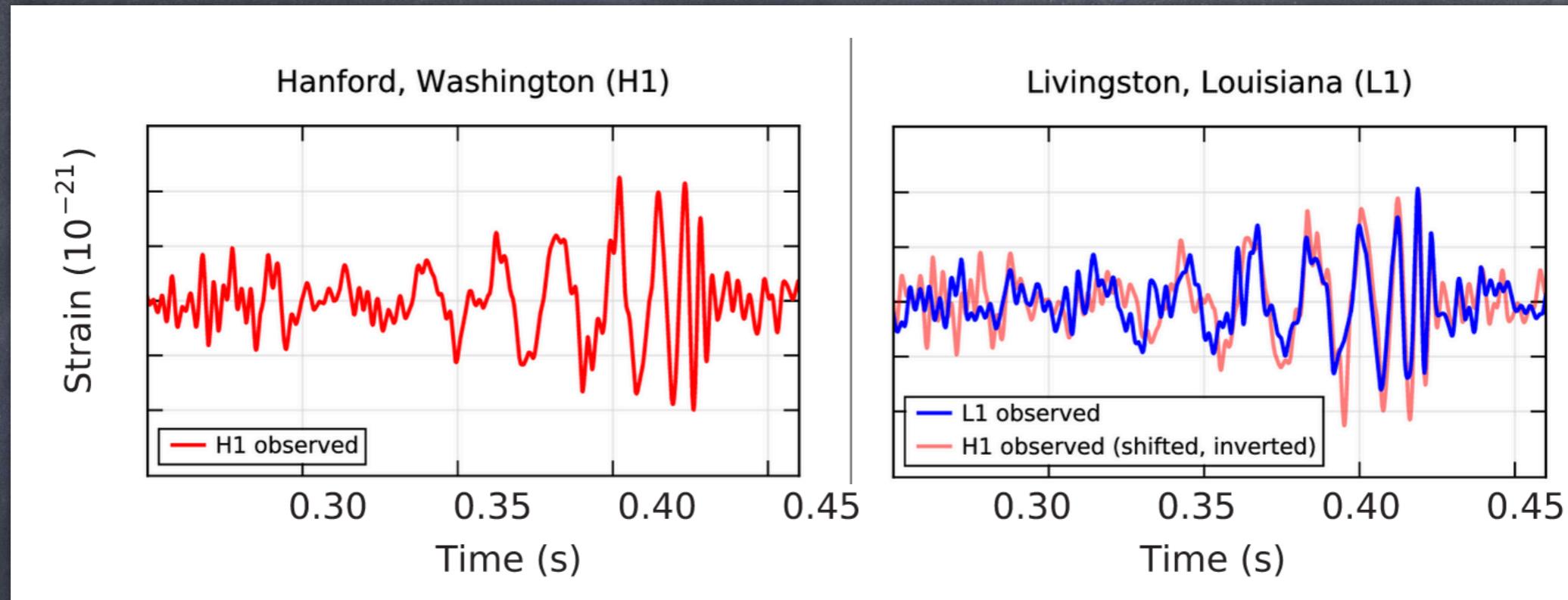
Updated 2020-09-02  
LIGO-Virgo | Frank Elavsky, Aaron Geller | Northwestern

# Masses in the Stellar Graveyard



LIGO-Virgo-KAGRA | Aaron Geller | Northwestern

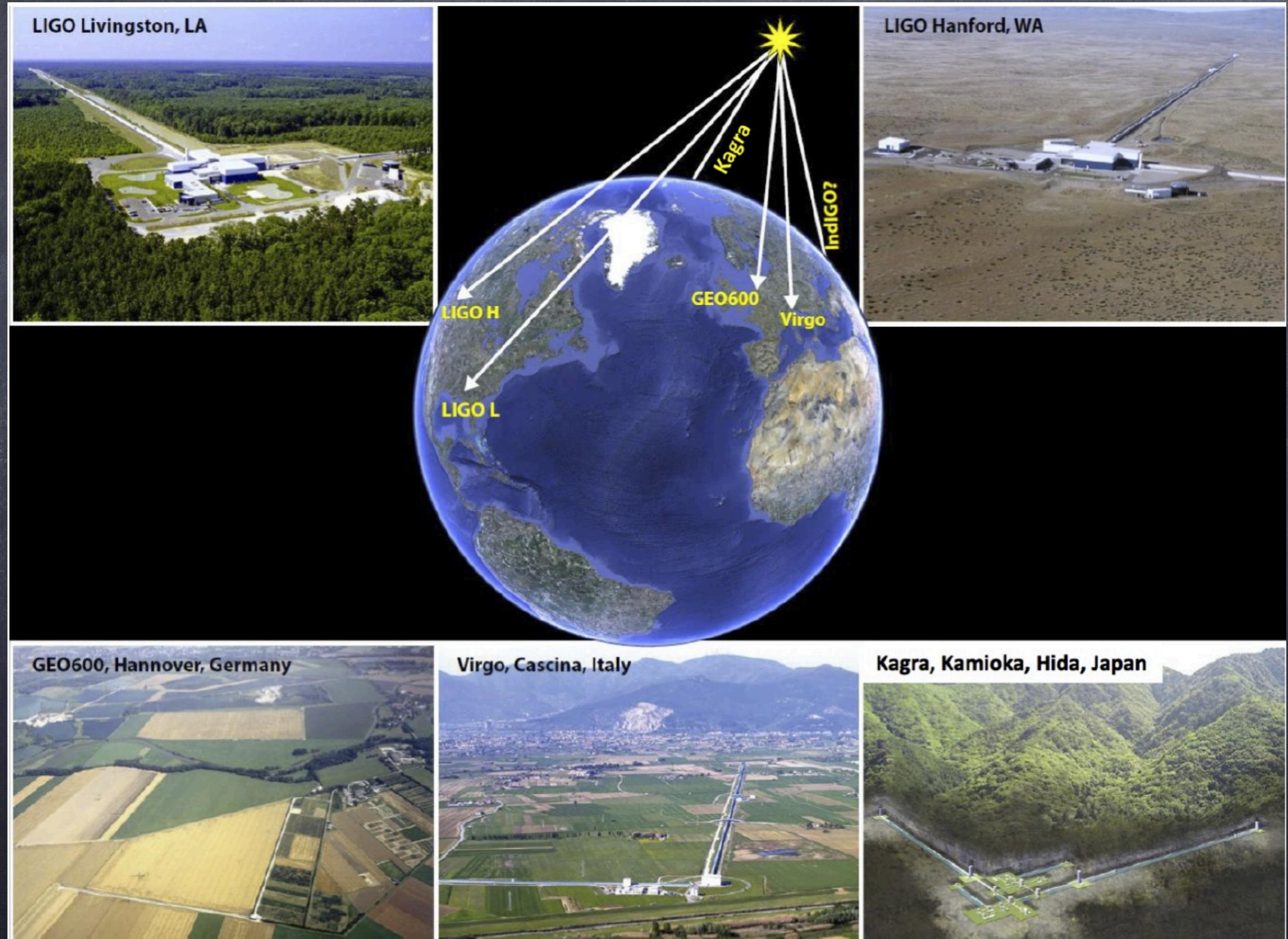
# GW150914



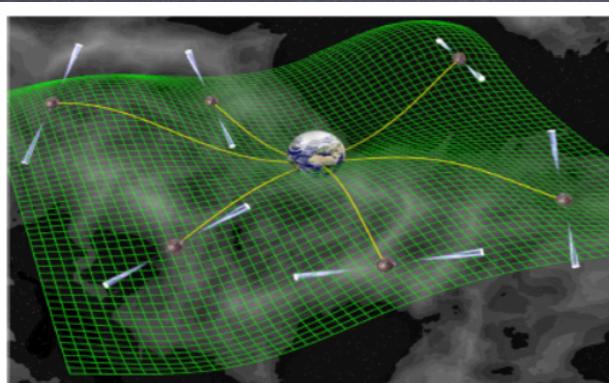
## Estimated source parameters

Quantity	Value	Upper/Lower error estimate	Unit
Primary black hole mass	<b>36.2</b>	+5.2 -3.8	M sun
Secondary black hole mass	<b>29.1</b>	+3.7 -4.4	M sun
Final black hole mass	<b>62.3</b>	+3.7 -3.1	M sun
Final black hole spin	<b>0.68</b>	+0.05 -0.06	
Luminosity distance	<b>420</b>	+150 -180	Mpc
Source redshift, $z$	<b>0.09</b>	+0.03 -0.04	
Energy radiated	<b>3.0</b>	+0.5 -0.5	M sun

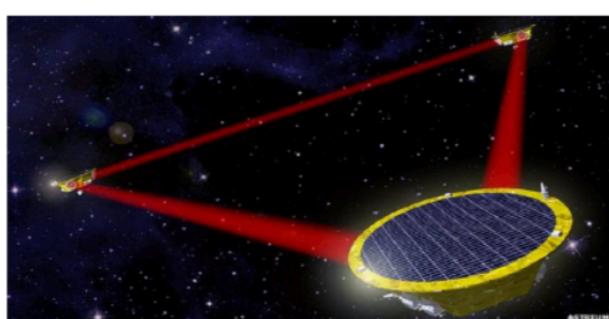
# Current GW detectors



# Upcoming detectors



Pulsar Timing Arrays



LISA

May 2022: Mission feasibility passed  
2037?: Science



Einstein Telescope

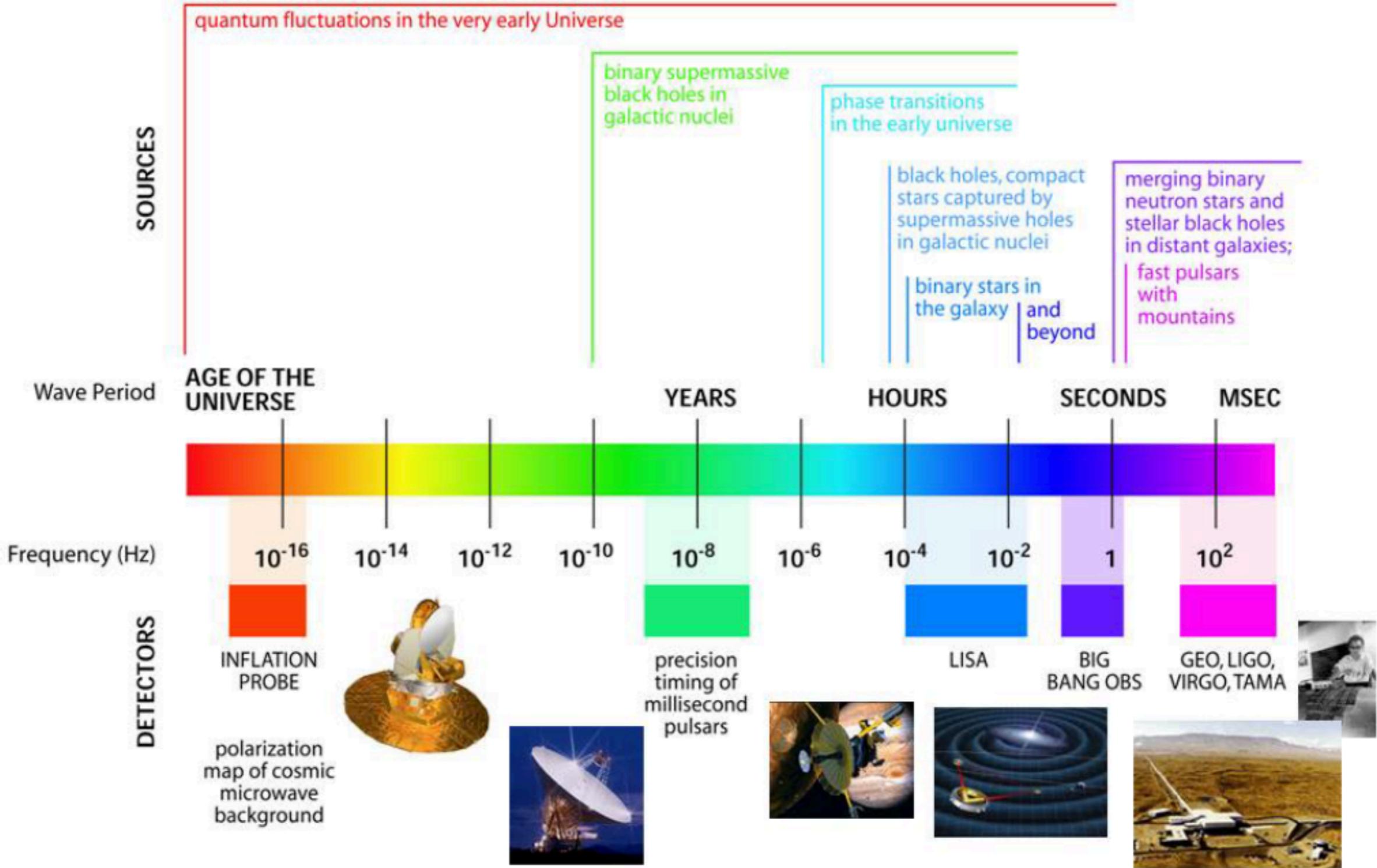
2024: Choice of site  
2035?: Science



Cosmic Explorer

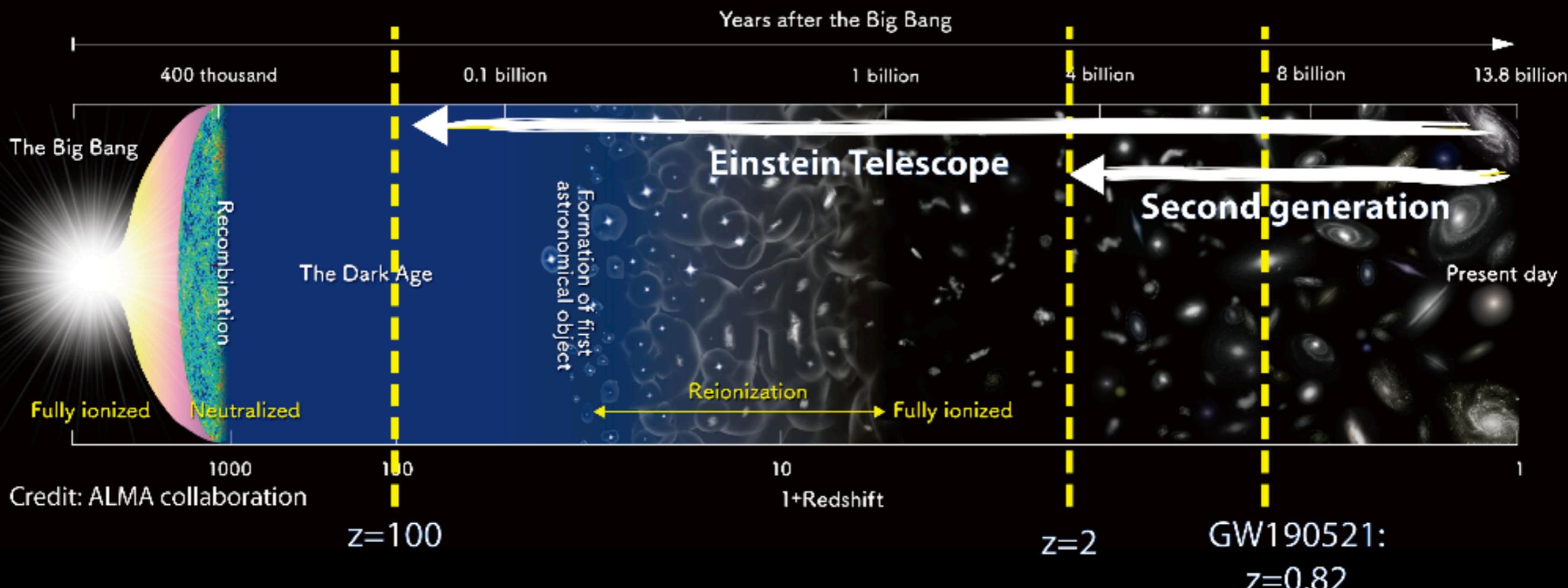
2035?: Science

# THE GRAVITATIONAL WAVE SPECTRUM

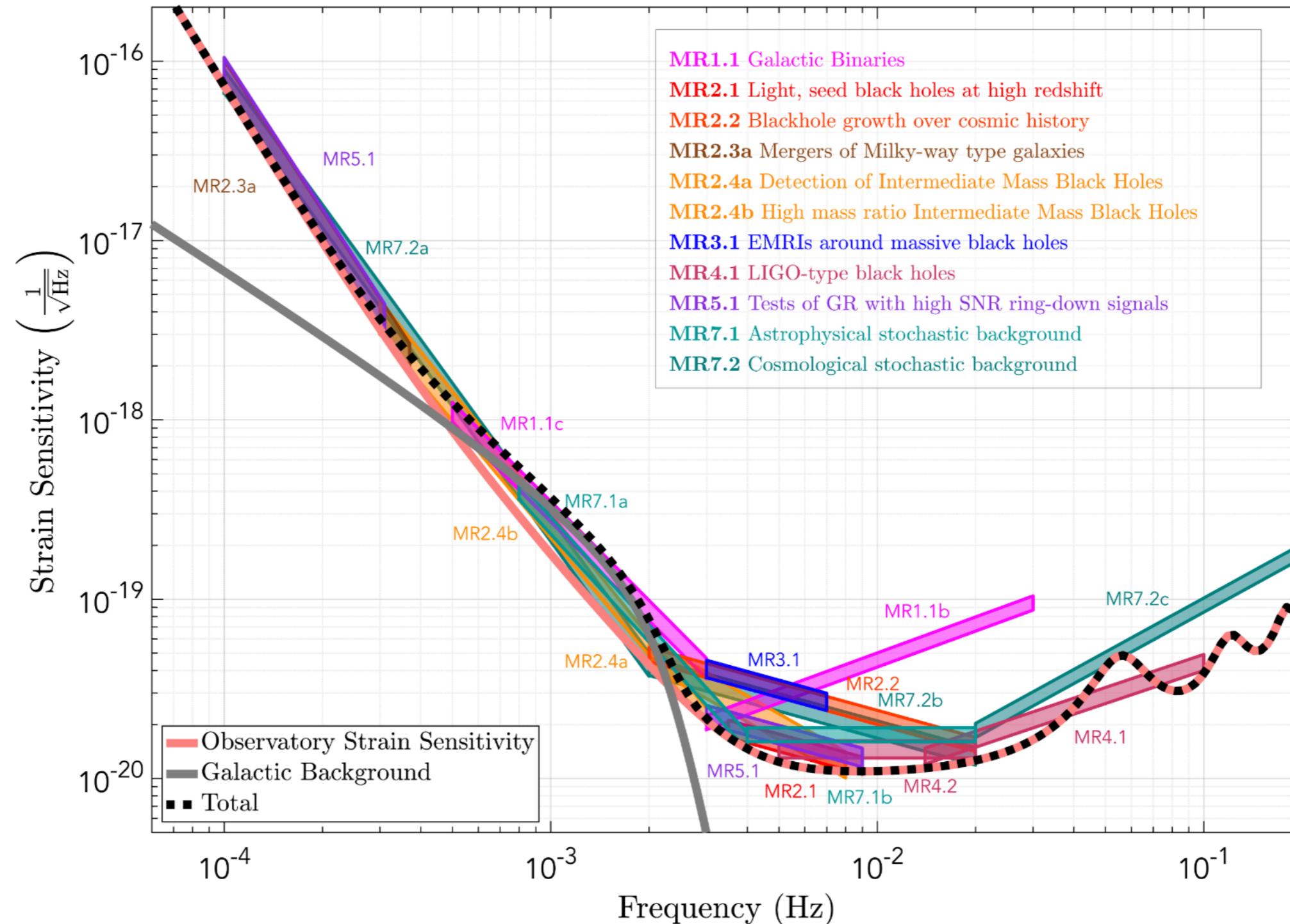


# 3G: Deep in the dark age

## Detection horizon for black-hole binaries



# LISA Science objectives



**Figure 2: Mission constraints on the sky-averaged strain sensitivity of the observatory for a 2-arm configuration (TDI X),  $\sqrt{S_h(f)}$ , derived from the threshold systems of each observational requirement.**

# GW and string theory

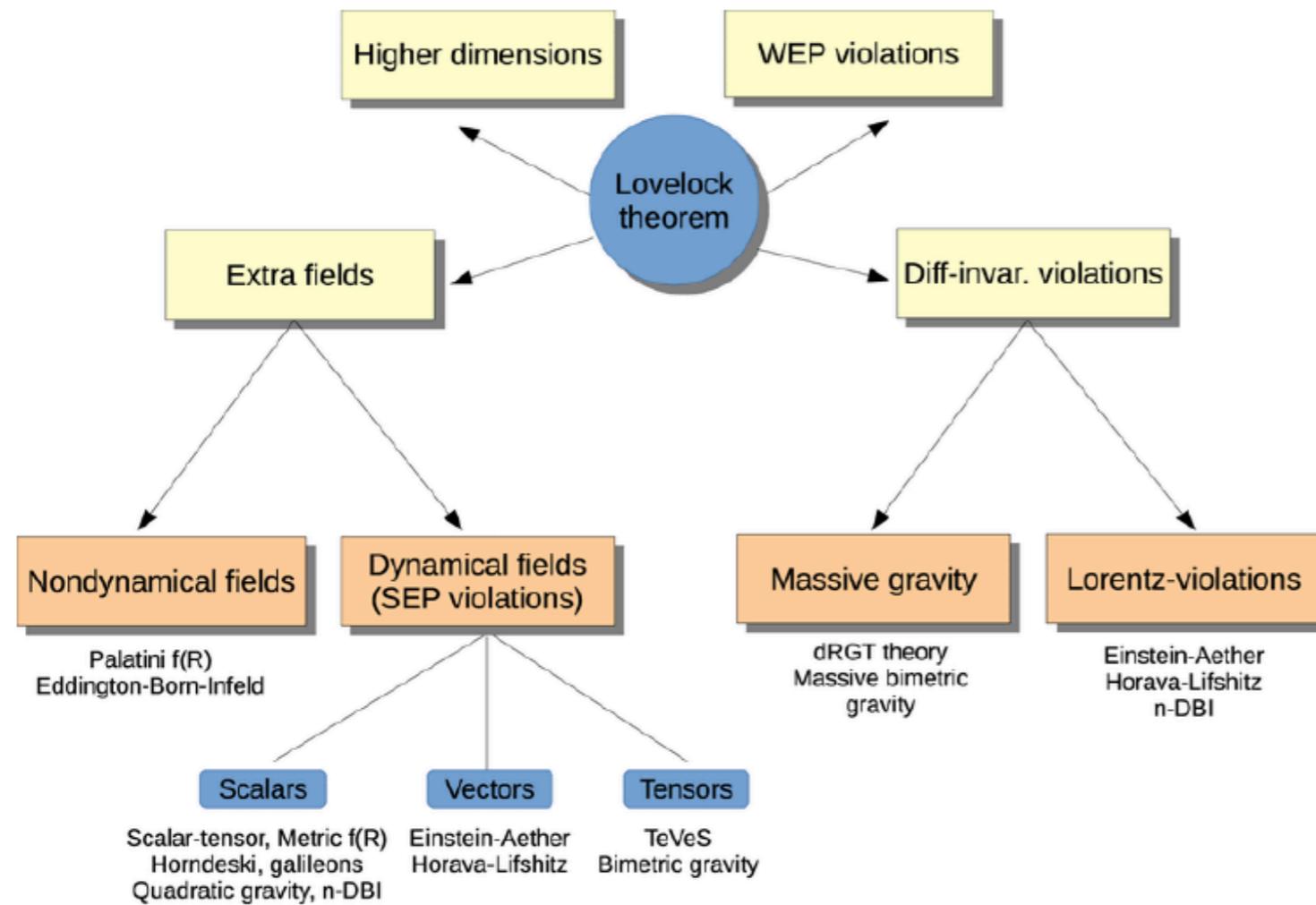
- (Huge) Gap between Planck scale and observations
- Effective field theory approach:  
GR+suppressed modifications
- Current and future observations:  
Strong field GR (no quantum, yet?)

# Modifications of GR

## ➤ Lovelock's theorem:

*"In four spacetime dimensions the only divergence-free symmetric rank-2 tensor constructed solely from the metric  $g_{\mu\nu}$  and its derivatives up to second differential order, and preserving diffeomorphism invariance, is the Einstein tensor plus a cosmological term."*

## ➤ Relaxing one or more of the assumptions allows for a plethora of alternative theories:



Berti et al., CQG 32, 243001 (2015)

# Diffeomorphic invariant higher curvature corrections start at $O(R^3)$

Leading order in the curvature expansion:  
the Einstein-Hilbert action

$$S_\Lambda + S_{\text{EH}} = \Lambda \int d^4x \sqrt{|g|} + \frac{1}{16\pi G} \int d^4x \sqrt{|g|} R.$$

Four terms at  $O(R^2)$ : (one is parity-odd)

$$\tilde{R}^{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta}^{\rho\sigma}.$$

$$S_{(2)} = \frac{\ell^2}{16\pi G} \int d^4x \sqrt{|g|} (b_1 R^2 + b_2 R_{\mu\nu} R^{\mu\nu} + b_3 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}) + b_4 \tilde{\mathcal{X}}_4$$

But two combinations are topological (Gauss-Bonnet and Pontryagin):

$$\mathcal{X}_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2,$$

$$\tilde{\mathcal{X}}_4 = R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}.$$

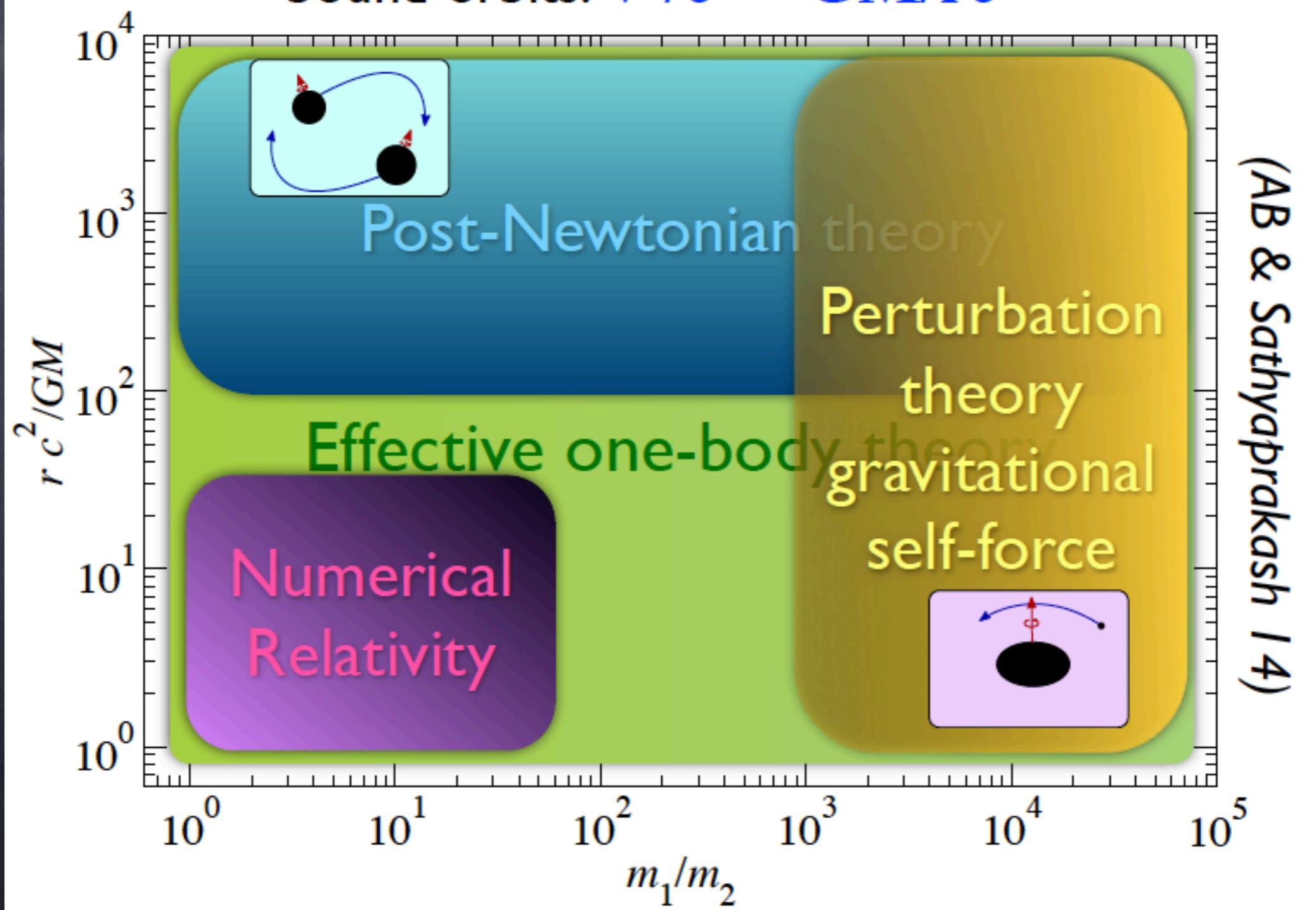
The other two parameters can be absorbed by a field-redefinition of the metric:

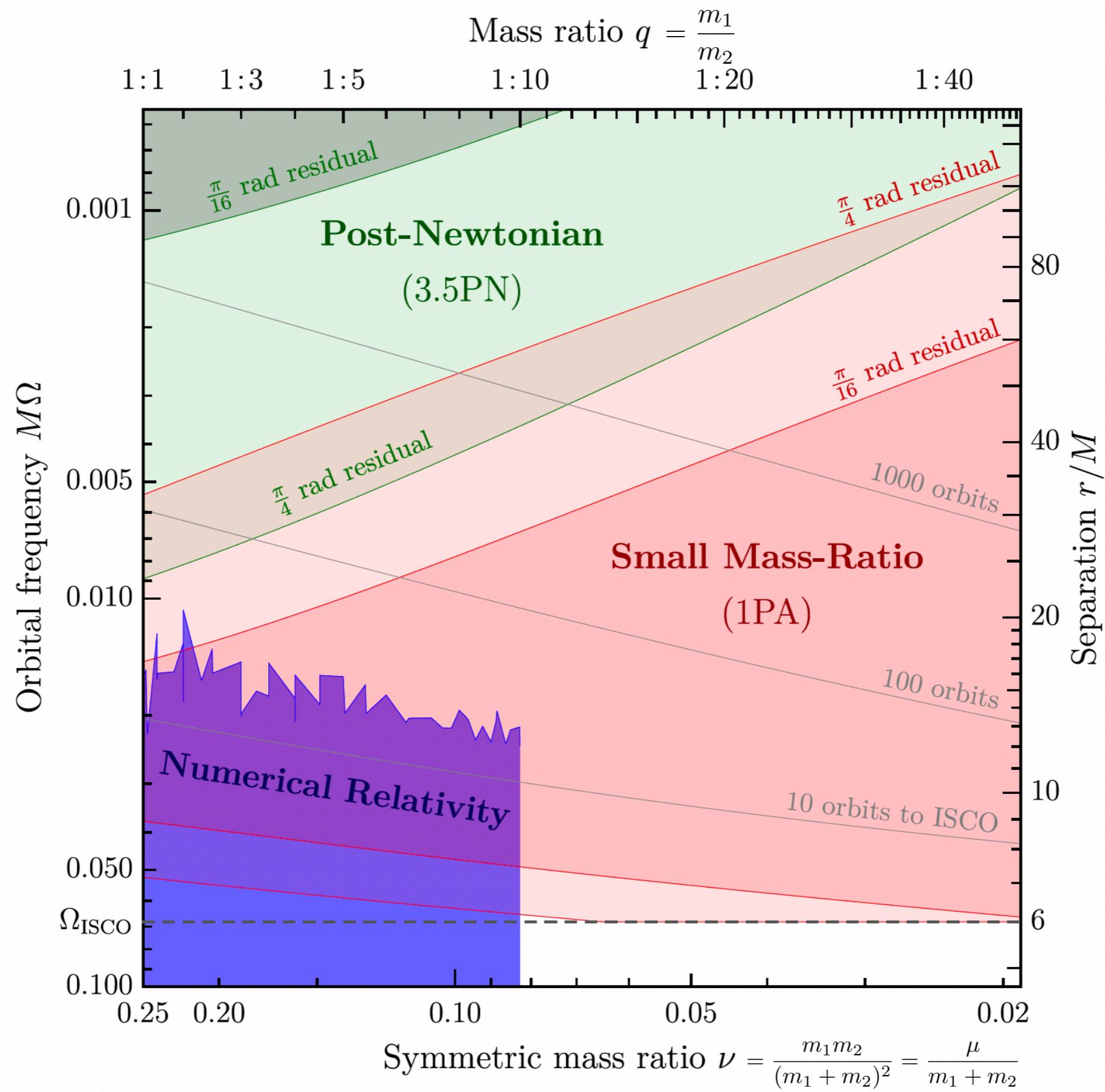
$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + Q_{\mu\nu}, \quad \mathcal{E}_{\mu\nu}^{(0)} = -\frac{1}{\sqrt{|g|}} \frac{\delta S^{(0)}}{\delta g^{\mu\nu}},$$

$$\sqrt{|g|} \mathcal{L}^{(0)}[g] = \sqrt{|\tilde{g}|} (\mathcal{L}^{(0)}[\tilde{g}] + \mathcal{E}_{\mu\nu}^{(0)} Q^{\mu\nu}) + \mathcal{O}(Q^2).$$

# These Lectures: Two-body problem in GR

bound orbits:  $v^2/c^2 \sim GM/rc^2$





[van de Meent, Pfeiffer, 2020]

# Outline of these lectures

## 1. Post-Newtonian/Post-Minkowskian theory

1.1. Propagation, linear GR theory

1.2. Interaction with test masses, freely falling frame, TT and detector frames, energy of GW

1.3. Generation of GW

1.4. Quasi-circular inspiral of compact binaries

1.5. Post-Newtonian corrections, the 1PN Einstein-Infeld-Hoffmann equations

1.6. Effective one-body resummation

## 2. Black hole Perturbation theory

2.1. Regge-Wheeler and Zerilli equations

2.2. Quasi-normal modes of Schwarzschild - Black Hole Spectroscopy

2.3. Newman-Penrose formalism, Petrov's classification, Teukolsky equation

2.4. Quasi-normal modes of Kerr

2.5. Mathisson-Papapetrou-Dixon theory

# References

1. M. Maggiore, Gravitational Waves, Volume 1. Sections 1, 3 and 4.
2. L. Blanchet, Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries, 1310.1528
3. N. Straumann, General Relativity
4. N. Deruelle, Mini-course on the 2-body problem in general relativity and scalar-tensor theories of gravity, ULB 2020.
5. L. Barack and A. Pound, Self-force and radiation reaction in general relativity, 1805.10385
6. A. Harte, Motion in classical field theories and the foundations of the self-force problem, 1405.5077
7. E. Berti, V. Cardoso, A. Starinets, 0905.2975
8. G. Compère, A. Fiorucci, 1801.07064

# 1.1. Propagation, Linear GR theory

# General Relativity

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R[g] + S_M$$



$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}$$

Minimal coupling     $\partial_\mu \rightarrow D_\mu$

$$T^{\mu\nu} = \frac{2c}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}}$$
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Diffeomorphism invariance

$$x^\mu \rightarrow x'^\mu(x)$$

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x(x'))$$

# Perturbation theory

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1$$

Residual gauge transformations:

$$x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x) \quad |\partial_\mu \xi_\nu| \ll 1$$

$$h_{\mu\nu}(x) \mapsto h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

Symmetries:

$$\xi_\mu = b_\mu \quad \rightarrow \quad h_{\mu\nu}$$
$$\xi_\mu = a_{[\mu\nu]} x^\nu \quad \text{invariant under Poincaré transformations}$$

# Linearized Riemann

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho})$$

Exercise: Prove it!

We define

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}$$

Trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

Note

$$\bar{h} \equiv \eta^{\mu\nu} \bar{h}_{\mu\nu} = h - 2h = -h$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

## Exercise

Prove

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho})$$

Remember

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma})$$

$$R^\alpha_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha_{\beta\delta} + \Gamma^\alpha_{\epsilon\gamma}\Gamma^\epsilon_{\beta\delta} - (\gamma \leftrightarrow \delta)$$

board

# Linearized Einstein

$$\square \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

We can use the gauge freedom to go to  
harmonic / de Donder gauge / Lorenz gauge

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

Indeed, under  $x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x)$

$$\bar{h}_{\mu\nu} \mapsto \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho)$$

$$\partial^\nu \bar{h}_{\mu\nu} \mapsto \partial^\nu \bar{h}_{\mu\nu} - \square \xi_\mu$$

If originally  $\partial^\nu \bar{h}_{\mu\nu} = f_\mu(x)$  we need to solve  $\square \xi_\mu = f_\mu(x)$   
to reach harmonic gauge. This is always possible.

# In harmonic gauge

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Acting with  $\partial^\nu$  we get by consistency  $\partial^\nu T_{\mu\nu} = 0$

- This equation is important both for generation of GW and for propagation

Outside sources,

$$\square \bar{h}_{\mu\nu} = 0 = \left(-\frac{1}{c^2} \partial_t^2 + \nabla^2\right) \bar{h}_{\mu\nu}$$



GW propagate at light speed

# $T\bar{T}$ gauge

For  $\xi^\mu$  harmonic  $\square\xi^\mu = 0$  then  $\xi_{\mu\nu} \equiv \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho$  is harmonic.

→ Out of the 10 components of  $\bar{h}_{\mu\nu}$  only 2 propagate as a wave.

We choose  $\xi^0$  such that  $\bar{h} = 0$

Proof:  $tr(\bar{h}_{\mu\nu} + \xi_{\mu\nu}) = \bar{h} + 2\partial_\alpha\xi^\alpha - 4\partial_\alpha\xi^\alpha$

Solve the ODE  $\partial_0\xi^0 + \partial_i\xi^i = \frac{1}{2}\bar{h}$  for  $\xi^0(x^0, x^i)$

Residual gauge parameter:  $\xi^0(x^i)$  harmonic

It implies

$$\bar{h}_{\mu\nu} = h_{\mu\nu}$$

# TT gauge

For  $\xi^\mu$  harmonic  $\square\xi^\mu = 0$  then  $\xi_{\mu\nu} \equiv \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho$  is harmonic.

→ Out of the 10 components of  $\bar{h}_{\mu\nu}$  only 2 propagate as a wave.

We choose  $\xi^i$  such that  $h^{0i} = 0$

Proof:  $h_{0i} + \xi_{0i} = h_{0i} + \partial_0\xi_i + \partial_i\xi_0$

Solve the ODE  $\partial_0\xi^i = -h_{0i} - \partial_i\xi_0$  for  $\xi^i(x^0, x^j)$

Residual gauge parameter:  $\xi^i(x^j)$  harmonic

It implies

$$h_{0i} = 0$$

Harmonic gauge reduces to

$$\mu = 0 \rightarrow \partial^0 h_{00} + \partial^i h_{0i} = \partial^0 h_{00} = 0 \quad \rightarrow \quad h_{00}(\vec{x})$$

$$\mu = i \rightarrow \partial^j h_{ij} = 0$$

Newtonian potential  
that is irrelevant for propagation  
We set it to zero

$$h_{0\mu} = 0, \quad h_{ii} = 0, \quad \partial^i h_{ij} = 0$$

We reached TT gauge. Note that it cannot be reached inside of sources because  $\square \bar{h}^{\mu\nu} \neq 0$

We write a decomposition in plane waves in terms of the polarisation tensor

$$h_{ij}^{TT}(x) = e_{ij}(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$k^\mu = \left( \frac{\omega}{c}, \vec{k} \right), \quad \frac{\omega}{c} = |\vec{k}|, \quad \hat{n} = \frac{\vec{k}}{|\vec{k}|}$$

# Polarized GW

Choose  $\hat{n}$  along the z axis

$$h_{ij}^{TT}(t, z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left[\omega\left(t - \frac{z}{c}\right)\right]$$

This gives the linear solution

$$\begin{aligned} ds^2 = & -c^2 dt^2 + dz^2 + \left(1 + h_+ \cos\left(\omega\left(t - \frac{z}{c}\right)\right)\right) dx^2 \\ & + \left(1 - h_+ \cos\left(\omega\left(t - \frac{z}{c}\right)\right)\right) dy^2 + 2h_\times \cos\left(\omega\left(t - \frac{z}{c}\right)\right) dxdy \end{aligned}$$

# TT Projector

Given a plane wave  $h_{\mu\nu}(x)$  propagating along the direction  $\hat{n}$  already in harmonic gauge but not yet in TT gauge, we reach TT gauge as

$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}$$

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \quad P_{ij} \equiv \delta_{ij} - n_i n_j$$

## Exercise

Prove the following properties of

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \quad P_{ij} \equiv \delta_{ij} - n_i n_j$$

(i) projector

$$\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$$

(ii) transverse

$$n^i\Lambda_{ij,kl} = 0 = n^k\Lambda_{ij,kl}$$

(iii) traceless

$$\Lambda_{ii,kl} = 0 = \Lambda_{ij,kk}$$

(iv) harmonic

$$\square h_{ij} = 0 = \partial^k \bar{h}_{kl} \Rightarrow \quad \square(\Lambda_{ij,kl} h_{kl}) = 0$$
$$\partial^j(\Lambda_{ij,kl} h_{kl}) = 0$$

## Exercise

Prove the following properties of

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \quad P_{ij} \equiv \delta_{ij} - n_i n_j$$

(i) projector

$$\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$$

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(iv) harmonic

$$\square h_{ij} = 0 = \partial^k \bar{h}_{kl} \Rightarrow \square(\Lambda_{ij,kl} h_{kl}) = 0$$

$$\partial^j(\Lambda_{ij,kl} h_{kl}) = 0$$

$$P_{ii} = 2$$

$$\square n_i = -2n_i$$

Hints:

$$\partial_a n_i \partial_b n_i = \gamma_{ab}$$

$$\partial_i \theta^a \partial_a n_j = P_{ij}$$

$$n_i P_{ij} = 0$$

board

1.2. Interaction with test  
masses, freely falling frame,  
 $\text{TT}$  and detector frames,  
energy of GW

# Interaction of GW with test masses

- Detectors idealized as test masses
- Computations done in a reference frame. Physics invariant under the choice
- GW are simple in TT gauge. It corresponds to a specific reference frame/observer
- Detector is more intuitive in another frame, the detector proper frame
- We need to switch between the two frames
- Physical intuition comes from the geodesic deviation equation

# Local inertial frame

- It is always possible to choose coordinates such that the Christoffel symbols vanish at one point
- The resulting coordinate system around that point is called a local inertial frame

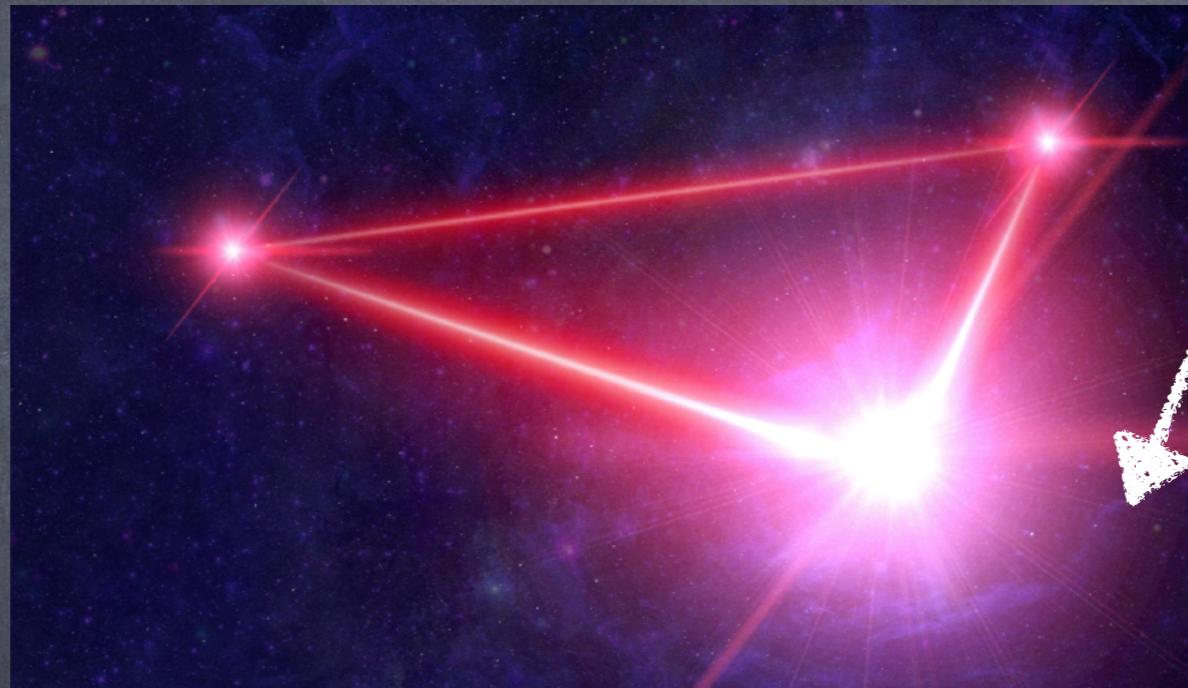
# Freely falling frame

- It is always possible to choose coordinates such that on an entire timeline geodesic, all Christoffel symbols vanish
- For each point  $P$  in this frame, the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2}|_P = 0.$$

- In this frame, a test mass is freely falling. It gives a realization of the equivalence principle.

# Laser Interferometer Space Antenna (LISA)



3 drag-free satellites:  
Spacecrafts that adjust their position with thrusters  
in order to remain centred about a freely floating mass

To be launched in 2034

# Fermi normal coordinates

A freely falling frame naturally define a special coordinate system around the test mass located at  $(t, 0, 0, 0)$ . It is called a Fermi normal coordinate system.

Since

$$\Gamma_{\beta\gamma}^{\alpha}|_{\gamma} = 0$$

the metric around the test mass has no linear term in  $x^i$

There are quadratic terms proportional to the Riemann tensor,

$$\begin{aligned} ds^2 \approx -c^2 dt^2 [1 + R_{0i0j} x^i x^j] - 2cdx^i dt \left( \frac{2}{3} R_{0jik} x^j x^k \right) \\ + dx^i dx^j \left[ \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l \right] \end{aligned}$$

# Geodesic equation

Consider a curve  $x^\mu(\lambda)$

The interval  $ds$  between 2 points separated by  $d\lambda$  is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda^2$$

The velocity is  $u^\mu = \frac{dx^\mu}{d\lambda}$

For a timelike curve the proper time  $\tau$  is defined as

$$c^2 d\tau^2 = -ds^2 = -g_{\mu\nu} u^\mu u^\nu d\lambda^2$$

or after using  $\lambda = \tau$   $g_{\mu\nu} u^\mu u^\nu = -c^2$

The classical trajectory of a particle test mass  $m$  is obtained by extremizing the action

$$S = -m \int_{\tau_i}^{\tau_f} d\tau \longrightarrow$$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\rho}^\mu(x) u^\nu u^\rho = 0$$

# Parallel transport

Consider a geodesics with proper time  $\tau$



We introduce the covariant derivative along the curve  $x^\mu(\tau)$

$$\frac{DV^\mu}{D\tau} \equiv \frac{\partial V^\mu}{\partial \tau} + \Gamma_{\nu\rho}^\mu V^\nu \frac{dx^\rho}{d\tau}$$

Property:  $\frac{DV^\mu}{D\tau}$  transforms as a vector.

The vector is parallelly transported along the curve.

# Geodesic deviation

Now consider 2 geodesics, each one with proper time  $\tau$



$$\frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma_{\nu\rho}^\mu(x + \xi) \frac{d(x^\nu + \xi^\nu)}{d\tau} \frac{d(x^\rho + \xi^\rho)}{d\tau} = 0$$

If  $|\xi^\mu|$  is negligible with respect to the variation of the gravitational field, we can expand at first order in  $\xi^\mu(\tau)$

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{d\xi^\rho}{d\tau} + \xi^\sigma \partial_\sigma \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

Exercise

Prove:

$$\frac{D^2\xi^\mu}{D\tau^2} = -\xi^\sigma R^\mu_{\nu\sigma\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau}$$

# Geodesic deviation

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{D}{D\tau} \left( \frac{\partial \xi^\mu}{\partial \tau} + \Gamma_{\nu\rho}^\mu \xi^\nu \frac{dx^\rho}{d\tau} \right)$$

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{\partial^2 \xi^\mu}{\partial \tau^2} + \frac{dx^\lambda}{d\tau} \partial_\lambda \Gamma_{\nu\rho}^\mu \xi^\nu \frac{dx^\rho}{d\tau} + \Gamma - terms$$

Use  $\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{d\xi^\rho}{d\tau} + \xi^\sigma \partial_\sigma \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$

$$\frac{D^2\xi^\mu}{D\tau^2} = -\xi^\sigma \partial_\sigma \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} + \frac{dx^\lambda}{d\tau} \partial_\lambda \Gamma_{\nu\rho}^\mu \xi^\nu \frac{dx^\rho}{d\tau} + \Gamma - terms$$

$$\frac{D^2\xi^\mu}{D\tau^2} = \xi^\sigma (\partial_\nu \Gamma_{\sigma\rho}^\mu - \partial_\sigma \Gamma_{\nu\rho}^\mu) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} + \Gamma - terms$$



$$\frac{D^2\xi^\mu}{D\tau^2} = -\xi^\sigma R_{\nu\sigma\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau}$$

# TT frame

This is the coordinate frame in which the metric is in TT gauge

Consider a test mass initially at rest at  $\tau = 0$ .

The geodesic equation is

$$\begin{aligned}\frac{d^2x^i}{d\tau^2}|_{\tau=0} &= -[\Gamma_{\nu\rho}^i(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}]|_{\tau=0} \\ &= -[\Gamma_{00}^i(\frac{dx^0}{d\tau})^2]\end{aligned}$$

because

$$\frac{dx^i}{d\tau}|_{\tau=0} = 0$$

At linear order,  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  after taking Cartesian coordinates for the background.

$$\Gamma_{\nu\rho}^{\mu} = \frac{1}{2}\eta^{\mu\sigma}(\partial_{\nu}h_{\rho\sigma} + \partial_{\rho}h_{\nu\sigma} - \partial_{\sigma}h_{\nu\rho})$$

$$\Gamma_{00}^i = \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00}) = 0$$

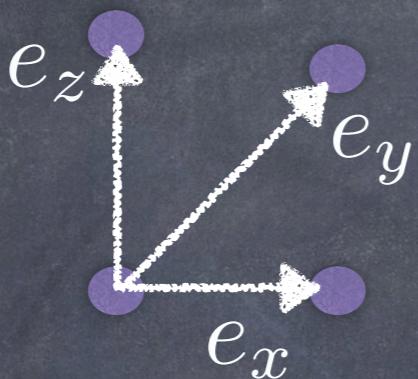
in TT gauge. Therefore,

$$\frac{d^2x^i}{d\tau^2}|_{\tau=0} = 0$$

In TT frame, particles which were at rest before the arrival of the wave remain at rest even after the arrival of the wave!  
(non-linear effects are highly subleading)

The coordinates stretch themselves so that the position of the free test masses do not change

We can define the coordinates using freely falling test masses



What about time? In TT gauge,  $h_{00} = h_{0i} = 0$

The proper time along a timelike trajectory  $x^\mu(\tau)$  is obtained from

$$c^2 d\tau^2 = c^2 dt^2 - (\delta_{ij} + h_{ij}^{TT}) \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} d\tau^2$$

For a test mass initially at rest,  $\frac{dx^i}{d\tau} = 0 \quad \forall \tau$

$$\rightarrow \tau = t$$

In TT gauge, the proper time of a free test mass initially at rest is the coordinate time

Of course, proper distances change!

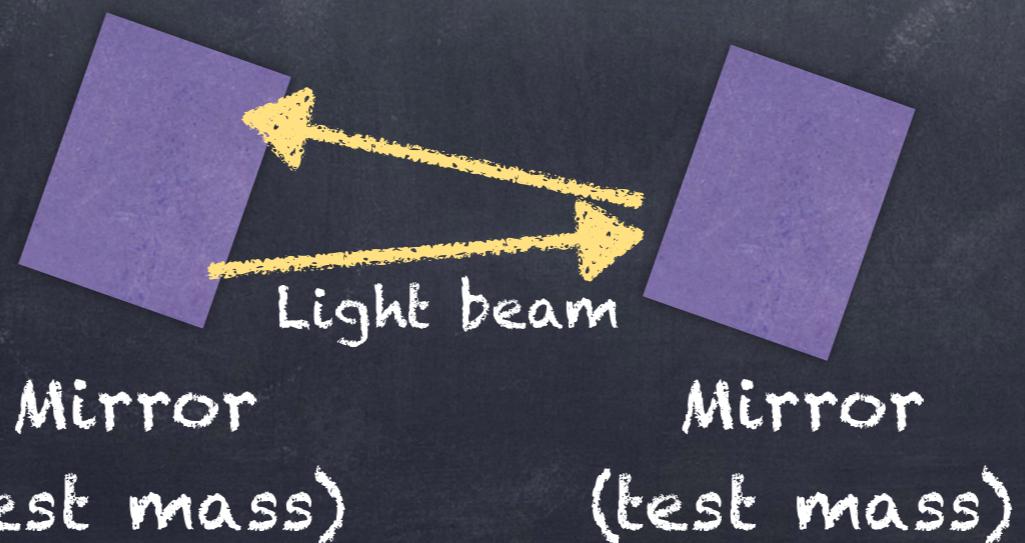
Consider the distance between  $(t, x_1, 0, 0)$  and  $(t, x_2, 0, 0)$

The coordinate distance  $L = x_2 - x_1$  is a constant

But the proper distance after the passage of the wave is

$$\begin{aligned}s &= \int_1^2 \sqrt{g_{xx}} dx = \sqrt{g_{xx}}(x_2 - x_1) \\&= \sqrt{1 + h_+ \cos(\omega t)} L \approx L\left(1 + \frac{1}{2}h_+ \cos(\omega t)\right)\end{aligned}$$

This is the basis of interferometry



# Earth detector frame (LIGO/Virgo)

- Distance computed using rigid rulers
- No free fall with respect to the local inertial frame

$\vec{a} = -\vec{g}$  is the acceleration of the laboratory

- Local rotation with respect to local gyroscopes (e.g. Foucault pendulum)
- $\vec{\Omega}$  is the angular velocity of the laboratory with respect to local gyroscopes.

# Earth detector frame (LIGO/Virgo)

Result up to  $O(r^2)$  is

$$r = \sqrt{x^i x^i}$$

inertial acc.	grav. redshift	Lorentz time dil.
$\frac{2}{c^2} \vec{a} \cdot \vec{x}$	$\frac{1}{c^4} (\vec{a} \cdot \vec{x})^2$	$-\frac{1}{c^2} (\vec{\Omega} \times \vec{x})^2 + R_{0i0j} x^i x^j$
$\frac{1}{c} \epsilon_{ijk} \Omega^j x^k$ Sagnac effect	$-\frac{2}{3} R_{0ijk} x^j x^k$ GW and varying grav.	
		GW and varying grav.
$dx^i dx^j [\delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l]$		
		GW and varying grav.

This is the detector frame used by experimentalists on Earth

We denote by  $L_B$  the typical variation scale of the metric

$$\frac{\vec{a}}{c^2} \sim \frac{1}{L_B}, \quad R_{\mu\nu\rho\sigma} \sim \frac{1}{L_B^2}$$

# Earth detector frame (LIGO/Virgo)

$$ds^2 \approx -c^2 dt^2 \left[ 1 + \frac{2}{c^2} \vec{a} \cdot \vec{x} + \frac{1}{c^4} (\vec{a} \cdot \vec{x})^2 - \frac{1}{c^2} (\vec{\Omega} \times \vec{x})^2 + R_{0i0j} x^i x^j \right] \\ + 2cdtdx^i \left[ \frac{1}{c} \epsilon_{ijk} \Omega^j x^k - \frac{2}{3} R_{0ijk} x^j x^k \right] \\ + dx^i dx^j \left[ \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l \right]$$

We denote by  $L_B$  the typical variation scale of the metric

$$\frac{\vec{a}}{c^2} \sim \frac{1}{L_B}, \quad R_{\mu\nu\rho\sigma} \sim \frac{1}{L_B^2}$$

At order  $O(\frac{r}{L_B})^0 \rightarrow$  Minkowski (in contrast to TT gauge where there is no such expansion!)

$O(\frac{r}{L_B})^1 \rightarrow$  Newtonian corrections

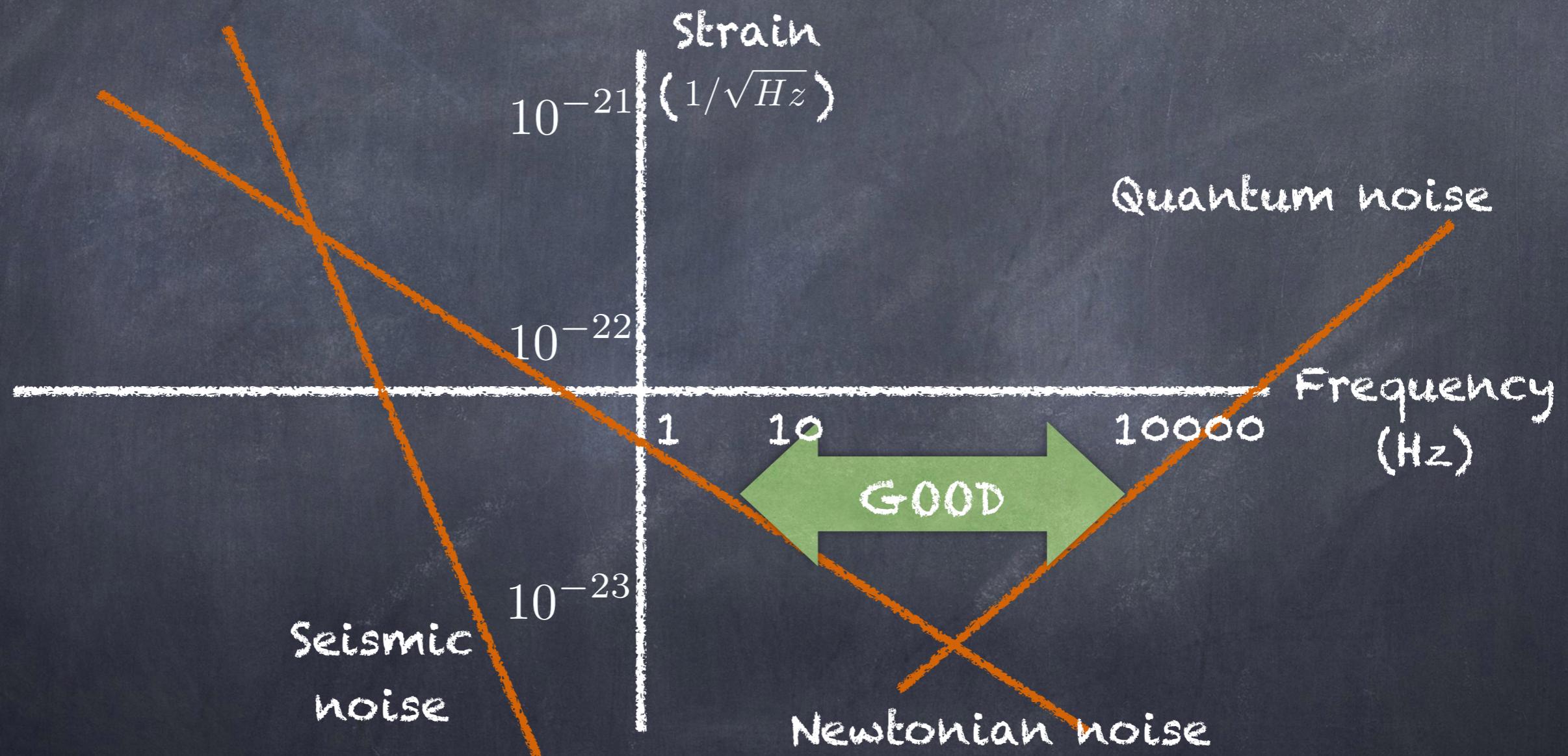
Geodesic equation:  $\frac{d^2 x^i}{d\tau^2} = -a^i - 2(\vec{\Omega} \times \vec{v})^i + O(x^i)$

Earth gravity	Coriolis force
---------------	----------------

$O(\frac{r}{L_B})^2 \rightarrow$  GW and noise from varying gravity

In conclusion, a description in terms of Newtonian concept of forces is possible in the Earth detector frame.

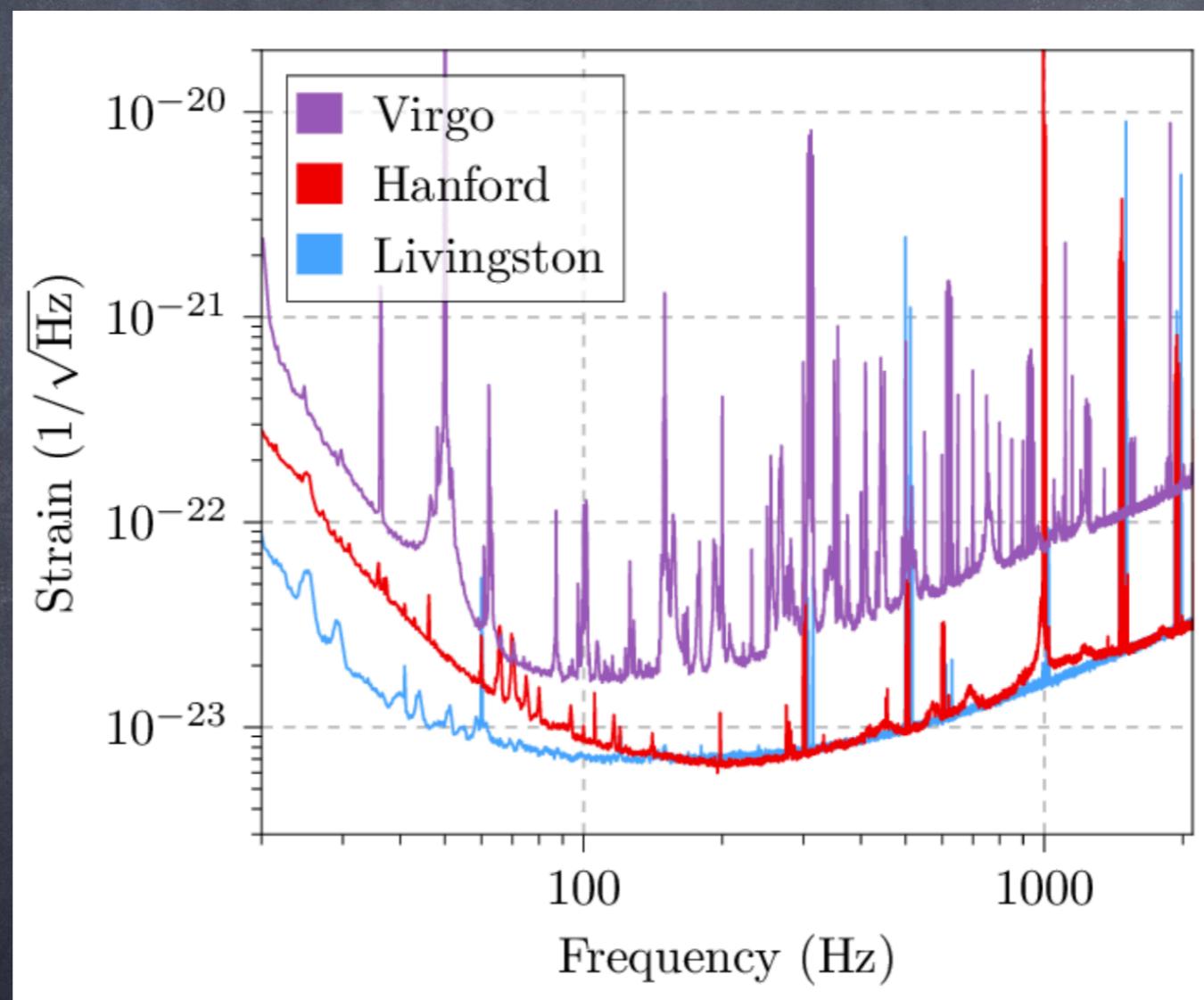
Gravitational Waves are subleading with respect to uncertainties in Earth gravity and local rotation, Newtonian and seismic noise.



Einstein Telescope: 2 possible sites: Sardaigna and Euregio

In conclusion, a description in terms of Newtonian concept of forces is possible in the Earth detector frame.

Gravitational Waves are subleading with respect to uncertainties in Earth gravity and local rotation, Newtonian and seismic noise.



To isolate GW, we focus on the detector in its frequency window. Acceleration is compensated by suspenders.

Only Riemann terms matter and the expression reduces to in the freely falling frame, as far as we only look at components of  $x^\mu(\tau)$  in the direction unconstrained by the suspension mechanism

If we look at the equation for geodesic deviation at the centre P of the local inertial frame

$$\Gamma_{\mu\nu}^\alpha|_P = 0 \quad \frac{D^2\xi^i}{D\tau^2} = \frac{d^2\xi^i}{d\tau^2} = -R^i_{0j0}\xi^j\left(\frac{dx^0}{d\tau}\right)^2$$

Non relativistic motion  $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \approx c$

$$\Rightarrow \ddot{\xi}^i = -c^2 R^i_{0j0} \xi^j$$

$$= -c^2 \left( -\frac{1}{2c^2} \ddot{h}_{ij}^{TT} \right) \xi^j$$



We can compute the Riemann in any frame including TT gauge.

Finally, in Earth detector frame in the directions unconstrained by suspenders:

$$\ddot{\xi}^i = \frac{1}{2} \ddot{h}_{ij}^{TT} \xi^j$$

In the Earth detector frame, the effect of GW on a point particle of mass  $m$  is described as a Newtonian force

$$F_i = \frac{m}{2} \ddot{h}_{ij}^{TT} \xi^j$$

without reference to GR.

Note that here we assumed  $|\xi^i| \ll L_B$  which is true for LIGO.

# Motion of test masses

- We consider a ring of test masses initially at rest in the Earth detector frame.
- We fix the origin at the centre of the ring
- Then  $\xi^i$  describes the distance w.r.t the origin (coordinate distance = proper distance)
- We consider GW propagating in the z direction. The ring is in the (x,y) plane

$$h_{ij}^{TT}(t, z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left[\omega\left(t - \frac{z}{c}\right)\right]$$

$$\ddot{\xi}^i = \frac{1}{2} \ddot{h}_{ij}^{TT} \xi^j$$

$$h_{ij}^{TT}(t, z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

↓  
sin

$$\ddot{\xi}^i = \frac{1}{2} \ddot{h}_{ij}^{TT} \xi^j$$

- If the particle is at  $z=0$  at  $t=0$  it will remain there.
- Therefore, GW are transverse in their physical effect: they displace the test masses transversally w.r.t. their direction of propagation

We write

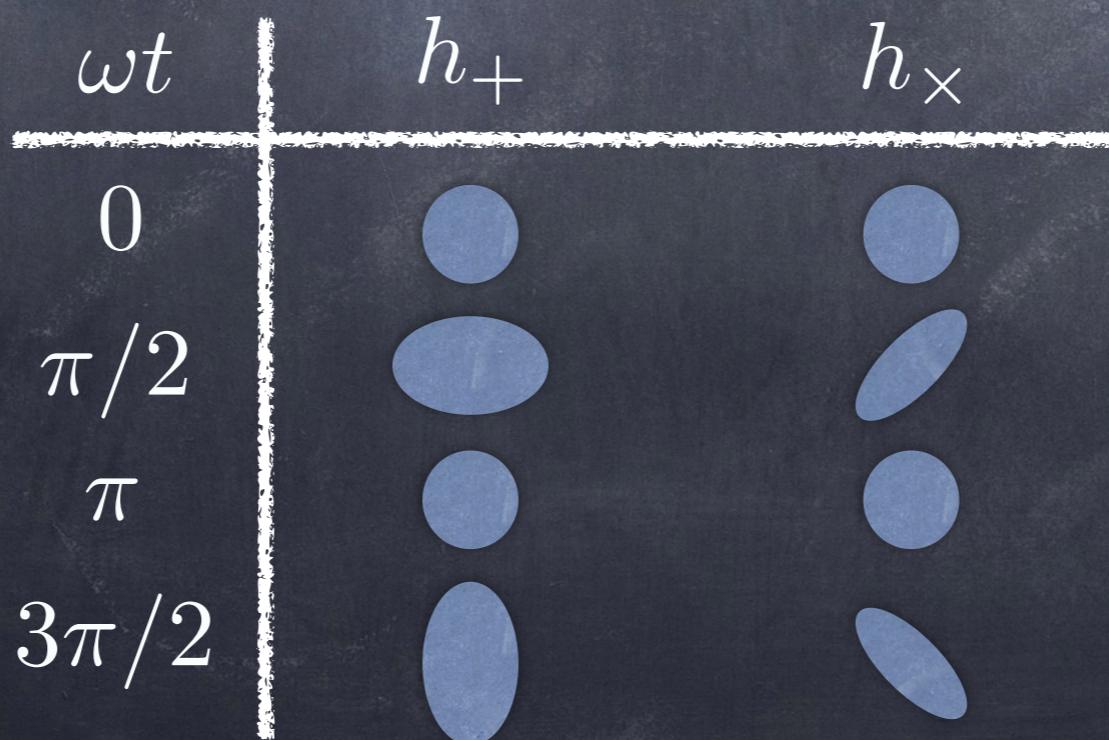
$$\xi_i(t) = (x_0 + \delta x(t), y_0 + \delta y(t), 0)$$

$(x_0, y_0)$  initial position

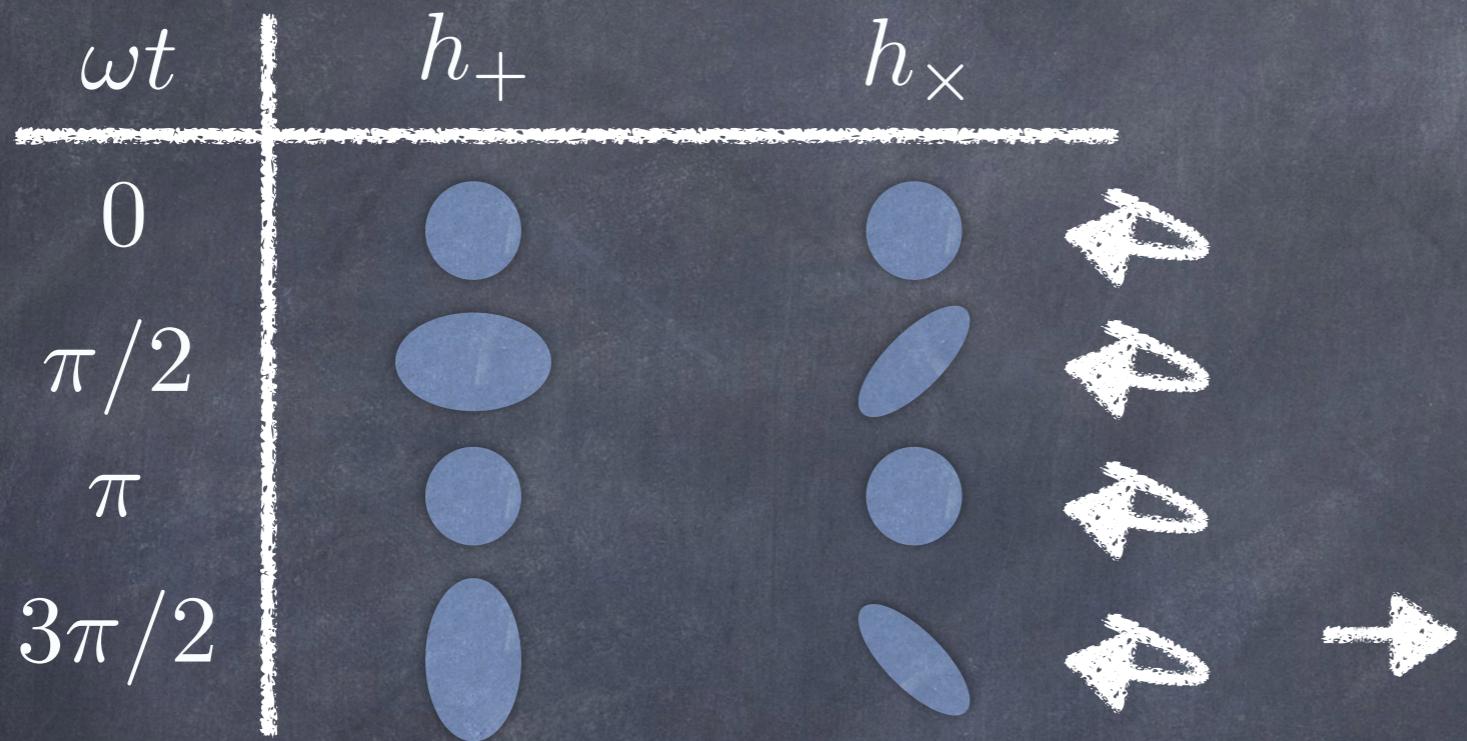
$$\delta x \ll x_0, \quad \delta y \ll y_0$$

For  $h_+$

$$\ddot{\delta x} = \frac{h_+}{2}(-\omega^2) \sin \omega t x_0 \quad \rightarrow \quad \delta x = \frac{h_+}{2} x_0 \sin \omega t$$
$$\ddot{\delta y} = \frac{h_+}{2} \omega^2 \sin \omega t y_0 \quad \rightarrow \quad \delta y = -\frac{h_+}{2} y_0 \sin \omega t$$



# Helicity of the graviton



Invariance under rotations of angle

$$\frac{2\pi}{2}$$

Graviton has helicity 2

Proof:

$$h_{\mu\nu}(x) \mapsto h'_{\mu\nu}(x') = \Lambda_\mu^\rho \Lambda_\nu^\sigma h_{\rho\sigma}(x)$$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi & 0 \\ 0 & \sin \psi & \cos \psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$h_{ij}^{TT}(t, z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})] \quad \mapsto \quad h'_{ij}^{TT}(t, z) = \begin{pmatrix} h'_+ & h'_\times & 0 \\ h'_\times & h'_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

$$h'_+ = h_+ \cos 2\psi - h_\times \sin 2\psi$$

$$h'_\times = h_+ \sin 2\psi + h_\times \cos 2\psi$$

$$h_\times \pm i h_+ \mapsto e^{\mp 2i\psi} (h_\times \pm i h_+)$$

helicity eigenstates

# Energy of GW

- It is clear that GW carry energy-momentum: they can accelerate masses !  
(Derived by Bondi in 1961)
- According to GR, any form of energy induces curvature
- GW then backreacts at second order beyond linear order and that curvature allows to define energy.

# Consistency of perturbation theory

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h_{\mu\nu}^{(2)} + \dots \quad |h_{\mu\nu}| \ll 1$$

Consistent if backreaction is smaller than the perturbation  
(far from masses)

In general, close to masses that generate GW:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + h_{\mu\nu}^{(2)} + \dots \quad |h_{\mu\nu}| \ll 1$$

We will distinguish the notion of background and perturbation by their frequency content :

background = low frequency  
perturbation = high frequency

We consider the situation in which in some reference frame we can separate the metric into a background plus fluctuations where separation is based on a scale in time or space ("short-wave expansion")

$$\bar{\lambda} \ll L_B$$

$$f \gg f_B$$



$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Two small parameters: (1)  $h \equiv O(h_{\mu\nu})$

$$(2) \frac{\bar{\lambda}}{L_B}, \frac{f_B}{f}$$

We now expand Einstein's equations to quadratic order in  $h_{\mu\nu}$

$$R_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$$

$$R_{\mu\nu}[g] = \bar{R}_{\mu\nu}[\bar{g}] + R_{\mu\nu}^{(1)}[h; \bar{g}] + R_{\mu\nu}^{(2)}[h, h; \bar{g}] + \dots$$

low freq modes	high freq modes	both modes
-------------------	--------------------	------------

$$e^{\vec{k} \cdot \vec{x}} e^{-\vec{k} \cdot \vec{x}} \sim e^0$$

Low mode eqs:  $\bar{R}_{\mu\nu} = -[R_{\mu\nu}^{(2)}]^{Low} + \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)^{low}$

High mode eqs:  $R_{\mu\nu}^{(1)} = -[R_{\mu\nu}^{(2)}]^{High} + \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)^{high}$

The low equation gives energy-momentum of GW. It can be obtained practically as follows. We introduce an intermediate time scale  $\bar{t}$

$$\frac{1}{f_B} \gg \bar{t} \gg \frac{1}{f}$$

We then average over  $\bar{t}$ , i.e. over many periods of GW:

$$\bar{R}_{\mu\nu} = -\langle R_{\mu\nu}^{(2)} \rangle + \frac{8\pi G}{c^4} \langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \rangle$$

This was understood in the sixties. This is a renormalisation group flow. We integrated out high frequencies to describe physics of low frequencies.

We define the effective stress-tensor of GW

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G} \langle R_{\mu\nu}^{(2)} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)} \rangle \quad R^{(2)} \equiv \bar{g}^{\mu\nu}R_{\mu\nu}^{(2)}$$

$$t = \bar{g}^{\mu\nu}t_{\mu\nu} = +\frac{c^4}{8\pi G} \langle R^{(2)} \rangle$$

because  $\bar{g}_{\mu\nu}$  is low frequency

We deduce

$$-\langle R_{\mu\nu}^{(2)} \rangle = \frac{8\pi G}{c^4} (t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t)$$

We define

$$\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \rangle \equiv \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}$$

The effective Einstein's equations at low frequencies

$$\bar{R}_{\mu\nu} = -\langle R_{\mu\nu}^{(2)} \rangle + \frac{8\pi G}{c^4} \langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \rangle$$

become

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}(\bar{T}_{\mu\nu} + t_{\mu\nu})$$

This is the coarse-grained form of Einstein's equations

We equated different orders in  $h$ . This is possible because there is a second small parameter  $\frac{\bar{\lambda}}{L_B} \ll 1$ .

Explicit expressions:

$$R_{\mu\nu}^{(1)} = \frac{1}{2} (\bar{D}^\alpha \bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha \bar{D}_\nu h_{\mu\alpha} - \bar{D}^\alpha \bar{D}_\alpha h_{\mu\nu} - \bar{D}_\nu \bar{D}_\mu h)$$

$$\begin{aligned} R_{\mu\nu}^{(2)} = & \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\alpha\beta} \left[ \frac{1}{2} \bar{D}_\mu h_{\rho\alpha} \bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\ & + h_{\rho\alpha} (\bar{D}_\nu \bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta \bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma}) \\ & \left. + \left( \frac{1}{2} \bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma} \right) (\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right] \end{aligned}$$

# Einstein equations' constraints on the small parameters

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}(\bar{T}_{\mu\nu} + t_{\mu\nu})$$

We choose coordinates such that  $\bar{g}_{\mu\nu} = O(1)$

$$\partial\bar{g}_{\mu\nu} \sim \frac{1}{L_B} \longrightarrow \bar{R}_{\mu\nu} \sim \partial^2\bar{g}_{\mu\nu} \sim \frac{1}{L_B^2}$$

$$\partial h_{\mu\nu} \sim \frac{h}{\bar{\lambda}} \longrightarrow R_{\mu\nu}^{(2)} \sim (\partial h)^2 \sim \frac{h^2}{\bar{\lambda}^2}$$

Einstein's equations give

$$\frac{1}{L_B^2} \sim \frac{h^2}{\bar{\lambda}^2} + (\text{matter}) \iff h \sim \frac{\bar{\lambda}}{L_B} \quad \text{if no matter}$$

$$\text{If dominant matter} \quad \frac{1}{L_B^2} \gg \frac{h^2}{\bar{\lambda}^2} \iff h \ll \frac{\bar{\lambda}}{L_B}$$

Therefore, if background slow frequency perturbation are neglected, the perturbation series in  $h$  breaks down.

# Gravitational energy-momentum tensor

The energy-momentum tensor can be computed assuming  
 $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$  (far from sources, ignoring Earth gravity)

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G} \left\langle R_{\mu\nu}^{(2)} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)} \right\rangle$$

We do the computation in harmonic and  $h = 0$  gauge

$$\partial_\mu h^{\mu\nu} = 0, \quad h = 0$$

In harmonic gauge, a perturbation is a superposition of plane waves

$$h_{\mu\nu} = \int \epsilon_{\mu\nu} e^{i\omega(t - \frac{\hat{n} \cdot \vec{x}}{c})}, \quad \vec{k} = \frac{\omega}{c} \hat{n}$$

We have

$$\begin{aligned} \partial_t h_{\mu\nu} &= i\omega h_{\mu\nu} \\ \partial_i h_{\mu\nu} &= -\frac{i\omega}{c} n_i h_{\mu\nu} \end{aligned} \quad \rightarrow \quad \partial_i h_{\mu\nu} = -\frac{1}{c} n_i \partial_t h_{\mu\nu}$$

Integrations by parts in  $R_{\alpha\beta}^{(2)}$  are possible

$\partial_t$  total derivatives drop because of time average

$\partial_i$  can be expressed as  $\partial_t$  because waves

$$\rightarrow \langle \partial_\alpha T_{\mu\nu}^\alpha \rangle = 0, \quad \forall T_{\mu\nu}^\alpha$$

## Exercise

Given

$$R_{\mu\nu}^{(2)} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta} \left[ \frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\ \left. + h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma}) \right. \\ \left. + \left(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma}\right)(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right]$$

Assume  $g_{\mu\nu} = \eta_{\mu\nu}$      $\partial_\mu h^{\mu\nu} = 0$ ,     $h = 0$      $\square h_{\alpha\beta} = 0$

Use  $\langle \partial_\alpha T_{\mu\nu}^\alpha \rangle = 0$ ,     $\forall T_{\mu\nu}^\alpha$  to prove

(i)  $\langle R_{(2)} \rangle = 0$

(ii)  $\langle R_{\mu\nu}^{(2)} \rangle = -\frac{1}{4} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$

## Solution

$$R_{\mu\nu}^{(2)} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta} \left[ \frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\ \left. + h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma}) \right. \\ \left. + \left( \frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma} \right) (\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right]$$

## Solution

$$R_{\mu\nu}^{(2)} = \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\alpha\beta} \left[ \frac{1}{2} \bar{D}_\mu h_{\rho\alpha} \bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\ \left. + h_{\rho\alpha} (\bar{D}_\nu \bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta \bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma}) \right. \\ \left. + \left( \frac{1}{2} \bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma} \right) (\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right]$$

$$\langle R^{(2)} \rangle = \frac{1}{2} \left\langle \frac{3}{2} \partial_\mu h^{\alpha\beta} \partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma} \partial^\beta h^{\alpha\gamma} + h^{\alpha\beta} \partial_\mu \partial^\mu h_{\alpha\beta} \right\rangle$$

## Solution

$$R_{\mu\nu}^{(2)} = \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\alpha\beta} \left[ \frac{1}{2} \bar{D}_\mu h_{\rho\alpha} \bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\ \left. + h_{\rho\alpha} (\bar{D}_\nu \bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta \bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma}) \right. \\ \left. + \left( \frac{1}{2} \bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma} \right) (\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right]$$

$$\langle R^{(2)} \rangle = \frac{1}{2} \left\langle \frac{3}{2} \partial_\mu h^{\alpha\beta} \partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma} \partial^\beta h^{\alpha\gamma} + h^{\alpha\beta} \partial_\mu \partial^\mu h_{\alpha\beta} \right\rangle$$

## Solution

$$R_{\mu\nu}^{(2)} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta} \left[ \frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\ \left. + h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma}) \right. \\ \left. + \left( \frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma} \right) (\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right]$$

$$\langle R^{(2)} \rangle = \frac{1}{2} \left\langle \frac{3}{2} \partial_\mu h^{\alpha\beta} \partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma} \partial^\beta h^{\alpha\gamma} + h^{\alpha\beta} \partial_\mu \partial^\mu h_{\alpha\beta} \right\rangle = 0$$

## Solution

$$\begin{aligned}
 R_{\mu\nu}^{(2)} = & \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\alpha\beta} \left[ \frac{1}{2} \bar{D}_\mu h_{\rho\alpha} \bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\
 & + h_{\rho\alpha} (\bar{D}_\nu \bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta \bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma}) \\
 & \left. + \left( \frac{1}{2} \bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma} \right) (\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right]
 \end{aligned}$$

$$\langle R^{(2)} \rangle = \frac{1}{2} \left\langle \frac{3}{2} \partial_\mu h^{\alpha\beta} \partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma} \partial^\beta h^{\alpha\gamma} + h^{\alpha\beta} \partial_\mu \partial^\mu h_{\alpha\beta} \right\rangle = 0$$

$$\begin{aligned}
 \langle R_{\mu\nu}^{(2)} \rangle = & \frac{1}{2} \left\langle \frac{1}{2} \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta} \partial^\alpha h_\nu^\beta - \partial_\alpha h_{\mu\beta} \partial^\beta h_\nu^\alpha \right. \\
 & \left. + h^{\alpha\beta} \partial_\mu \partial_\nu h_{\alpha\beta} + h^{\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu} - h^{\alpha\beta} \partial_\alpha \partial_\mu h_{\nu\beta} - h^{\alpha\beta} \partial_\alpha \partial_\nu h_{\mu\beta} \right\rangle
 \end{aligned}$$

## Solution

$$\begin{aligned}
 R_{\mu\nu}^{(2)} = & \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\alpha\beta} \left[ \frac{1}{2} \bar{D}_\mu h_{\rho\alpha} \bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\
 & + h_{\rho\alpha} (\bar{D}_\nu \bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta \bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma}) \\
 & \left. + \left( \frac{1}{2} \bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma} \right) (\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right]
 \end{aligned}$$

$$\langle R^{(2)} \rangle = \frac{1}{2} \left\langle \frac{3}{2} \partial_\mu h^{\alpha\beta} \partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma} \partial^\beta h^{\alpha\gamma} + h^{\alpha\beta} \partial_\mu \partial^\mu h_{\alpha\beta} \right\rangle = 0$$

$$\begin{aligned}
 \langle R_{\mu\nu}^{(2)} \rangle = & \frac{1}{2} \left\langle \frac{1}{2} \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta} \partial^\alpha h_\nu^\beta - \partial_\alpha h_{\mu\beta} \partial^\beta h_\nu^\alpha \right. \\
 & \left. + h^{\alpha\beta} \partial_\mu \partial_\nu h_{\alpha\beta} + h^{\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu} - h^{\alpha\beta} \partial_\alpha \partial_\mu h_{\nu\beta} - h^{\alpha\beta} \partial_\alpha \partial_\nu h_{\mu\beta} \right\rangle
 \end{aligned}$$

## Solution

$$\begin{aligned}
 R_{\mu\nu}^{(2)} = & \frac{1}{2} \bar{g}^{\rho\sigma} \bar{g}^{\alpha\beta} \left[ \frac{1}{2} \bar{D}_\mu h_{\rho\alpha} \bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \right. \\
 & + h_{\rho\alpha} (\bar{D}_\nu \bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta \bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma}) \\
 & \left. + \left( \frac{1}{2} \bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma} \right) (\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}) \right]
 \end{aligned}$$

$$\begin{aligned}
 \langle R^{(2)} \rangle = & \frac{1}{2} \left\langle \frac{3}{2} \partial_\mu h^{\alpha\beta} \partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma} \partial^\beta h^{\alpha\gamma} + h^{\alpha\beta} \partial_\mu \partial^\mu h_{\alpha\beta} \right\rangle = 0 \\
 \langle R_{\mu\nu}^{(2)} \rangle = & \frac{1}{2} \left\langle \frac{1}{2} \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta} \partial^\alpha h_\nu^\beta - \partial_\alpha h_{\mu\beta} \partial^\beta h_\nu^\alpha \right. \\
 & \left. + h^{\alpha\beta} \partial_\mu \partial_\nu h_{\alpha\beta} + h^{\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu} - h^{\alpha\beta} \partial_\alpha \partial_\mu h_{\nu\beta} - h^{\alpha\beta} \partial_\alpha \partial_\nu h_{\mu\beta} \right\rangle \\
 \langle R_{\mu\nu}^{(2)} \rangle = & -\frac{1}{4} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle
 \end{aligned}$$

# Gravitational energy -momentum tensor

Energy-momentum tensor is

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G} \langle R_{\mu\nu}^{(2)} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)} \rangle \quad \longrightarrow \quad t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$$

Do residual gauge transformation change the stress-tensor?

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$\delta_\xi t_{\mu\nu} = ?$$

## Exercise

Prove  $\delta_\xi t_{\mu\nu} = 0$

Using  $t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

## Solution

Prove  $\delta_\xi t_{\mu\nu} = 0$

Using  $t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Remember  $h = 0 \rightarrow \partial_\alpha \xi^\alpha = 0$   
 $\partial_\alpha h^{\alpha\mu} = 0 \rightarrow \square \xi^\mu + \partial_\mu \partial_\alpha \xi^\alpha = 0$

Therefore  $\langle \partial_\alpha (\partial_\dots h_{\mu\nu} \partial_\dots \xi^\sigma) \rangle = 0$

## Solution

Prove  $\delta_\xi t_{\mu\nu} = 0$

Using  $t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Remember  $h = 0 \rightarrow \partial_\alpha \xi^\alpha = 0$

$$\partial_\alpha h^{\alpha\mu} = 0 \rightarrow \square \xi^\mu + \partial_\mu \partial_\alpha \xi^\alpha = 0$$

Therefore  $\langle \partial_\alpha (\partial_\dots h_{\mu\nu} \partial_\dots \xi^\sigma) \rangle = 0$

Compute  $\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu (\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha) \partial_\nu h^{\alpha\beta} \rangle + (\mu \leftrightarrow \nu)$

$$\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu \partial_\alpha \xi_\beta \partial_\nu h^{\alpha\beta} \rangle + (\mu \leftrightarrow \nu)$$

$$\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu \xi_\beta \partial_\alpha \partial_\nu h^{\alpha\beta} \rangle + (\mu \leftrightarrow \nu) \sim 0$$

# Gravitational energy-momentum tensor

Energy-momentum tensor is

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G} \langle R_{\mu\nu}^{(2)} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)} \rangle \quad \longrightarrow \quad t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$$

Question: do residual gauge transformation change the stress-tensor? Answer: no.

Therefore, we can replace  $h_{\mu\nu}$  by  $h_{\mu\nu}^{TT}$

In particular, the effective energy density is

$$t^{00} = \frac{c^2}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle = \frac{c^2}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \quad \dot{f} = \partial_t f = c\partial_0 f$$

$$h_{ij}^{TT} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

Bianchi identities imply

$$\bar{D}^\mu (\bar{T}_{\mu\nu} + t_{\mu\nu}) = 0$$

This proves that one can associate to each Poincaré symmetry a conserved quantity (up to higher order perturbations)

Far from the sources

$$\partial^\mu t_{\mu\nu} = 0$$

→  $\int_V d^3x [\partial_0 t^{00} + \partial_i t^{i0}] = 0$  on a given volume  $V$

The effective energy of the volume is

$$E_V = \int d^3x t^{00}$$

$$\frac{1}{c} \frac{dE_V}{dt} = - \int_V d^3x \partial_i t^{0i} = - \int dA n_i t^{0i}$$

Take  $S$  a spherical surface  $dA = r^2 d\Omega$

$$\frac{dE_V}{dt} = -c r^2 \int d\Omega t^{0r}$$

↙

$$= \frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{TT} \frac{\partial}{\partial r} h_{ij}^{TT} \rangle$$

Single plane waves are not realistic as global solutions at large distances. Instead, a GW propagating radially outwards has the following form at large distances from the source (proven later)

$$h_{ij}^{TT} = \frac{1}{r} f_{ij} \left( t - \frac{r}{c} \right) + O(r^{-2})$$

This is similar to electromagnetic waves, up to a spin-2 polarization

## Exercise

Prove that  $h_{ij}^{TT} = \frac{1}{r} f_{ij} \left( t - \frac{r}{c} \right)$  obeys  $\square h_{ij}^{TT} = O(r^{-2})$

## Solution

Prove that  $h_{ij}^{TT} = \frac{1}{r} f_{ij} \left( t - \frac{r}{c} \right)$  obeys  $\square h_{ij}^{TT} = O(r^{-2})$

Use  $\square \Phi = \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu \Phi \right)$

$$ds^2 = -c^2 dt^2 + dr^2 \quad \sqrt{|g|} = c$$

$$ds^2 = -c^2 du^2 - 2cdudr \quad \sqrt{|g|} = c$$

Drop  $c$

$$x^\mu = (u, r) \quad g^{\mu\nu} = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

$$V^\mu = g^{\mu\nu} \partial_\nu \left( \frac{f_{ij}(u)}{r} \right) = \left( -\frac{-1}{r^2} f_{ij}, -\left(\frac{1}{r}\right) \partial_u f_{ij} + \left(-\frac{1}{r^2}\right) f_{ij} \right)$$

$$\partial_\mu V^\mu = \frac{1}{r^2} \partial_u f_{ij} + \frac{1}{r^2} \partial_u f_{ij} + \frac{1}{r^3} f_{ij} = O(r^{-2})$$

$$\frac{\partial}{\partial r} h_{ij}^{TT} = -\frac{1}{r^2} f_{ij} \left( t - \frac{r}{c} \right) + \frac{1}{r} \left( -\frac{1}{c} \right) \frac{\partial}{\partial t} f_{ij} = -\partial_0 h_{ij}^{TT} = \partial^0 h_{ij}^{TT}$$

→  $t^{0r} = t^{00}$

An observer sitting at large distances sees a plane wave front.

Since  $E_V$  decreases, GW carry away an energy flux

$$\begin{aligned} \frac{dE}{dAdt} &= +ct^{00} = \frac{c^3}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle \\ &= \frac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \end{aligned}$$

This is Einstein's formula for the flux-balance law of energy, with a factor of 2 corrected by Eddington.

# Poincaré flux-balance Laws

For each Killing vector of Minkowski, we have a conserved current :

$$J^\mu = t^{\mu\nu} \bar{\xi}_\nu \quad \partial_\mu J^\mu = 0$$

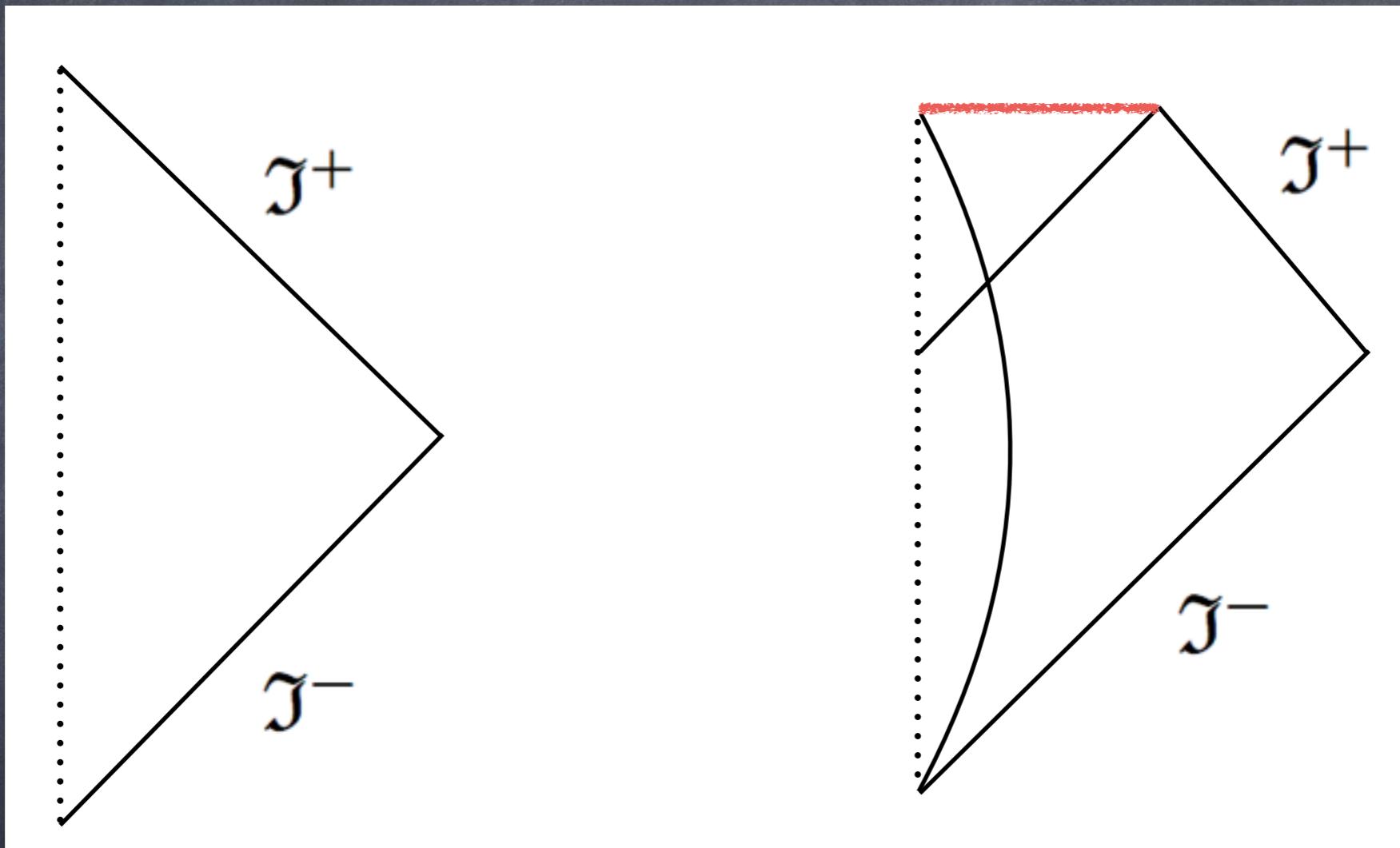
We can define the flux of momentum

$$\frac{dP^k}{dt} = -\frac{c^3}{32\pi G} r^2 \int d\Omega \langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \rangle$$

Also, the flux of angular momentum and centre of mass

Those are the flux-balance Laws for Poincaré charges

# Asymptotic view on flux-balance Laws



Approach of Bondi, van den Burg, Metzner, Sachs (BMS), 1962

# Einstein's solution

Radiative gauge [Newman-Unti gauge]

$$g_{rr} = -1, \quad g_{ru} = g_{r\theta} = g_{r\phi} = 0$$

The asymptotic solution takes the form

$$\begin{aligned} ds^2 &= -c^2 du^2 - 2cdudr + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \\ &+ \frac{2m}{r} du^2 + rC_{AB}dx^A dx^B + \dots \\ &+ \frac{N_A}{r} dudx^A + \dots \\ &+ \frac{1}{r^i} E_{AB}^{(i)} dx^A dx^B + \dots \end{aligned}$$

It depends on

$$C_{AB}, \quad m, N_A, E_{AB}^{(i)}, \quad i = 1, 2, \dots$$

Unconstrained

They obey flux-balance laws

# BMS flux-balance Laws

Supermomentum

$$Q_T(u) \equiv \int_S d^2\Omega m(u, \theta, \phi) T(\theta, \phi)$$

Super-Lorentz charge

$$Q_R(u) \equiv \int_S d^2\Omega N_A(u, \theta, \phi) R^A(\theta, \phi)$$

Higher spin charges

$$Q_S^{(i)}(u) \equiv \int_S d^2\Omega E_{AB}^{(i)}(u, \theta, \phi) S^{AB}(\theta, \phi)$$

$\partial_u Q_T(u)$  = flux on  $\mathfrak{I}^+$  dictated by Einstein's equations

$\partial_u Q_R(u)$  = flux on  $\mathfrak{I}^+$  dictated by Einstein's equations

$\partial_u Q_S^{(i)}(u)$  = flux on  $\mathfrak{I}^+$  dictated by Einstein's equations

## 1.3. Generation of GW

# Binaries as sources of GW

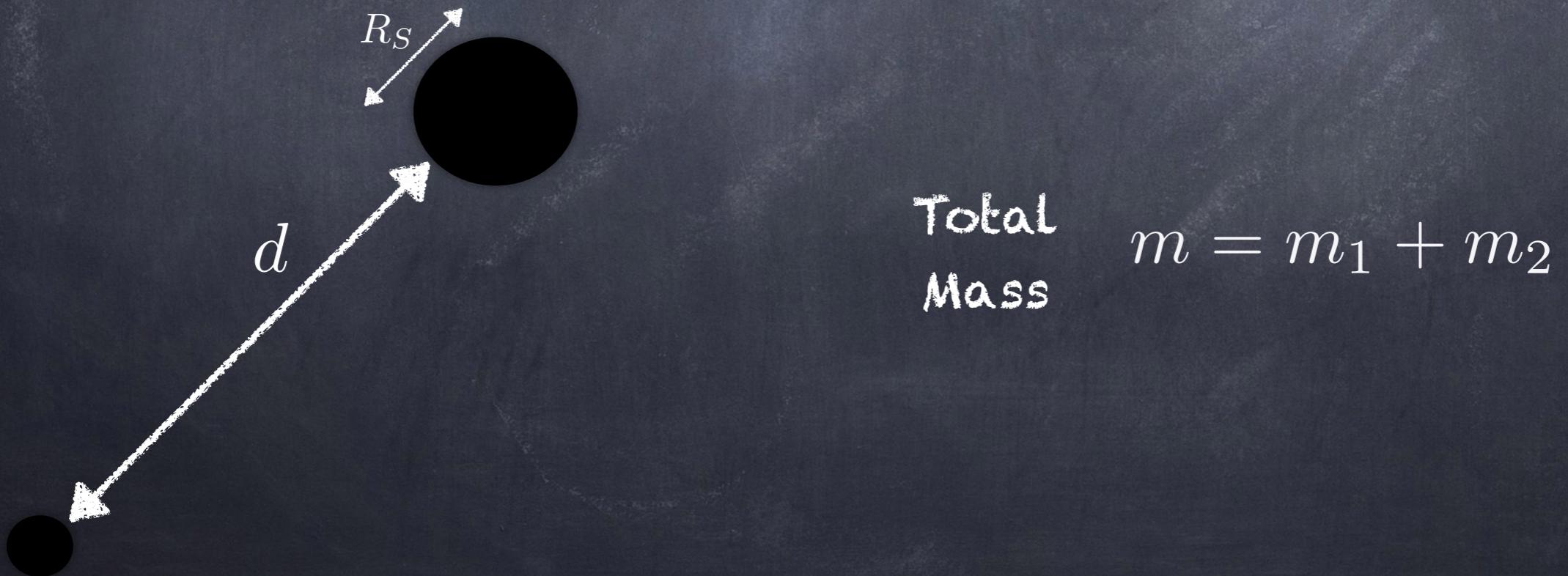
We consider a slowly moving source, which is weakly self-gravitating

$$\frac{\text{Typical velocity}}{c} \ll 1$$

$$\frac{R_S}{d} \ll 1$$

Schwarzschild radius of the source  
 $R_S = \frac{2Gm}{c^2}$  ← Total mass  
Size of the source

Typical example: a binary system in the early inspiral phase



Binaries admit only one small parameter  
as a result of gravitational binding

$$\frac{v}{c} \ll 1 \quad \frac{R_S}{d} \ll 1$$

Reduced Mass  $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Total Mass  $m = m_1 + m_2$

Gravitational binding gives

$$\frac{1}{2} \mu v^2 \sim \frac{1}{2} \frac{G \mu m}{d}$$

$$\frac{v^2}{c^2} \sim \frac{R_S}{d}$$

As a result, for binary compact objects, corrections in  $v/c$  induce corrections in  $G$ .

# GW wavelength in terms of the source size

- If  $\omega_s$  is the typical frequency of motion inside the source and  $d$  is the source size, the typical velocities are
- The frequency of radiation will also be of the order of

$$\omega_{gw} \sim 2\omega_s$$

(Proof: later on !)

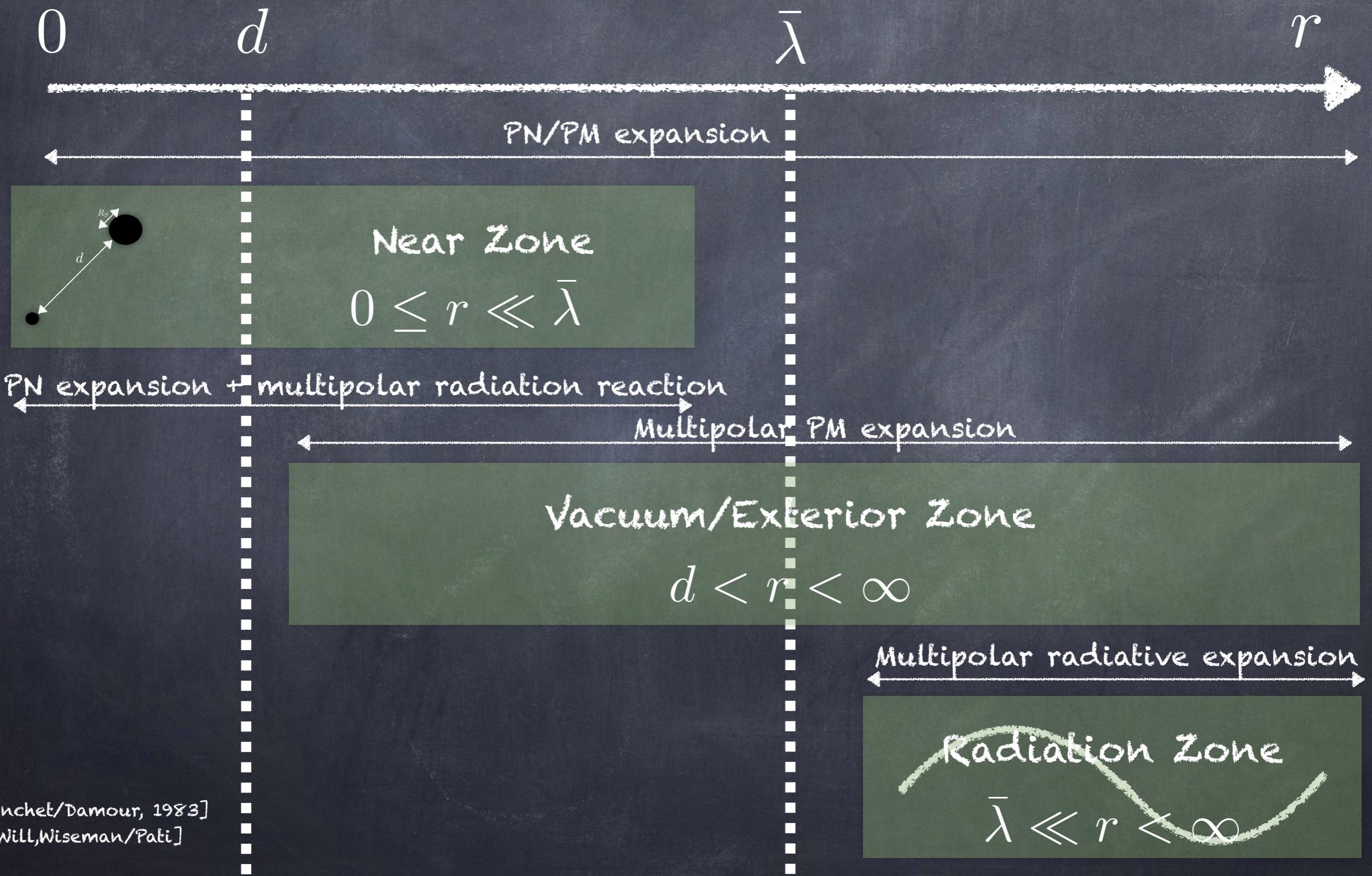
→ The reduced wavelength:  $\bar{\lambda} = \frac{c}{\omega_{gw}} \sim \frac{c}{\omega_s} \sim \frac{c}{v} d$

For a non-relativistic system,  $v \ll c$



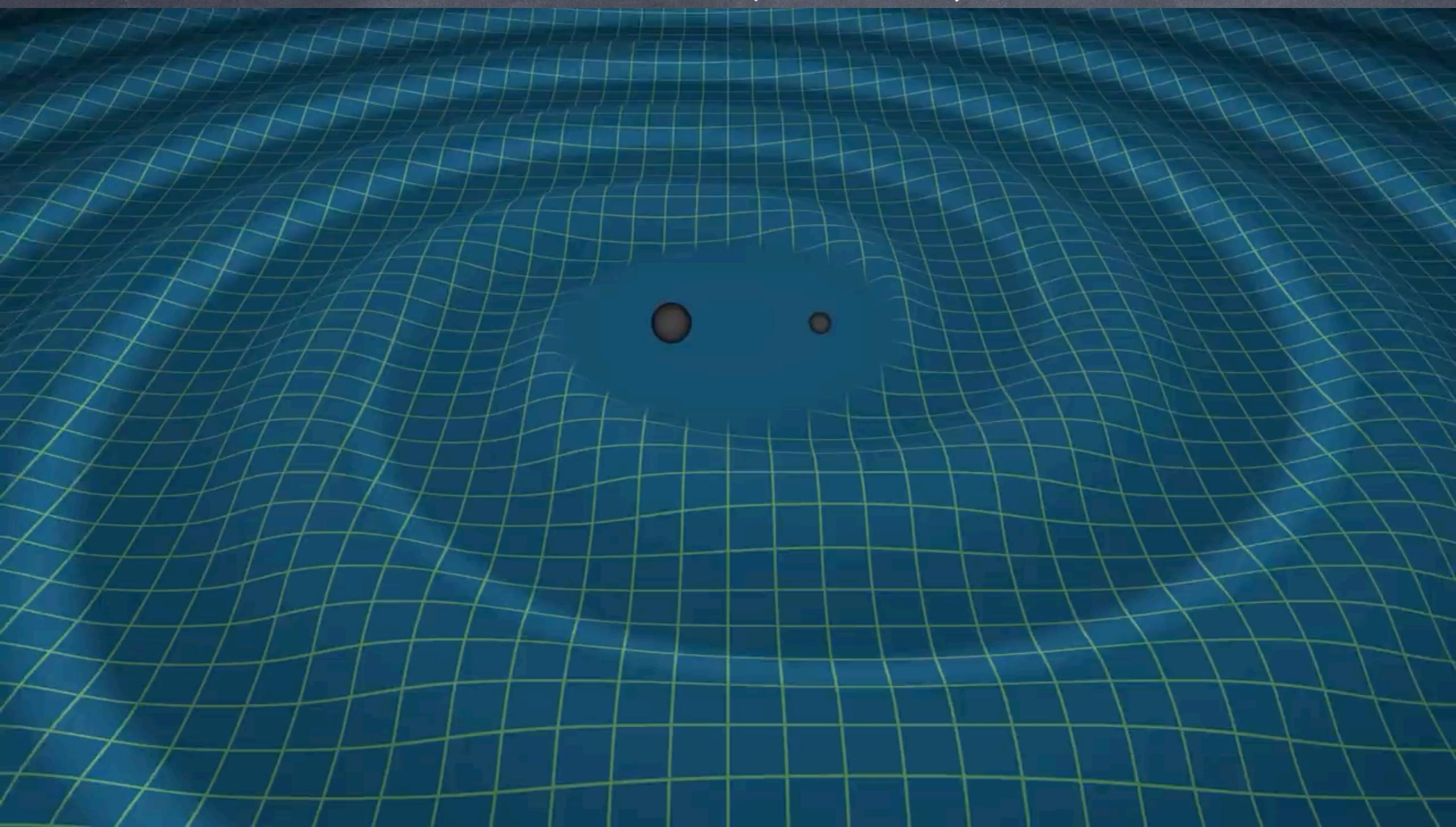
$$d \ll \bar{\lambda}$$

# The 3 zones of the Post-Newtonian/ Post-Minkowskian formalism



Multipolar PM expansion

PN expansion + multipolar radiation reaction



In the following, we will derive the motion at the lowest order:

- in the Near-Zone: Newtonian (OPN) + 2.5PN radiation-reaction
- in the exterior zone: Minkowski + Linear GW (1PM)

# Weak sources with arbitrary velocity

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\partial^\mu \bar{h}_{\mu\nu} = 0, \quad \partial^\nu T_{\mu\nu} = 0$$

We can solve it in terms of the retarded Green function:

$$\square G(x - x') = \delta^4(x - x')$$

$$\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} \int d^4x' G(x - x') T_{\mu\nu}(x')$$

Explicitly,

$$G(x - x') = \frac{-1}{4\pi |\vec{x} - \vec{x}'|} \delta(x_{ret}^0 - x'^0_{ret})$$

$$x'^0 = ct' \quad x^0_{ret} = ct_{ret} \quad t_{ret} = t - \frac{|\vec{x} - \vec{x}'|}{c}$$

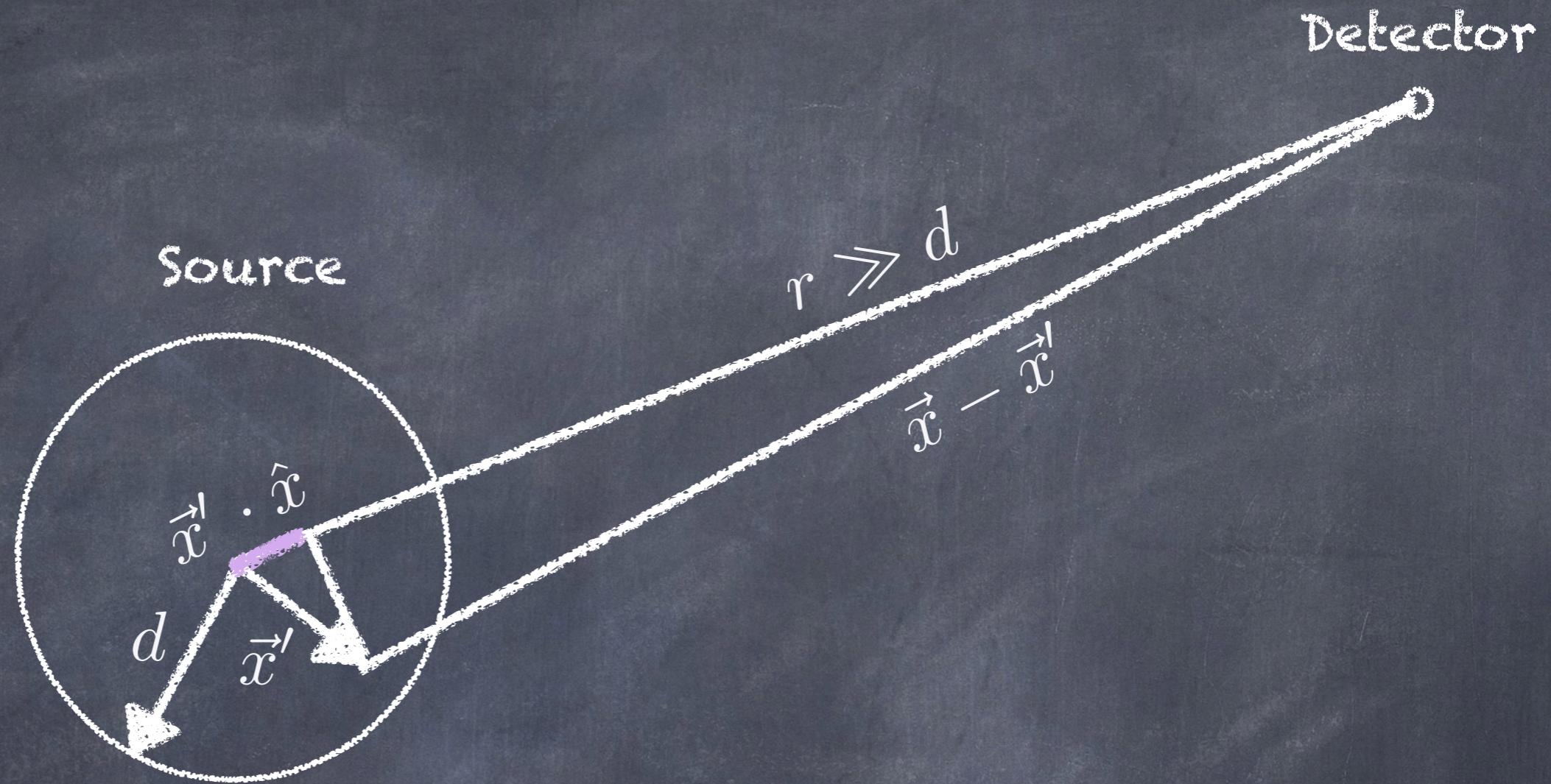
Recall that outside the source, we can use TT gauge and we have

$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}$$

Therefore, outside the source we have

$$h_{ij}^{TT}(t, \vec{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\vec{x}) \int d^3x' \frac{1}{|x - x'|} T_{kl}\left(t - \frac{|x - x'|}{c}, x'\right)$$

Note that  $h_{ij}^{TT}$  depends upon the integrals of the spatial components of  $T_{kl}$   
(temporal components are related by conservation)



$$|\vec{x} - \vec{x}'| = r - \vec{x}' \cdot \hat{x} + O(d^2/r), \quad \hat{x} \equiv \frac{\vec{x}}{r}$$

Therefore, for large  $r$ ,

$$h_{ij}^{TT} = \frac{1}{r} \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{x}) \int d^3x' T_{kl}\left(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c}, \vec{x}'\right)$$

# Weak sources with low velocity

Fourier transform:

$$T_{kl}\left(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c}, \vec{x}'\right) = \int \frac{d^4 k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \vec{k}) e^{-i\omega(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c}) + i\vec{k} \cdot \vec{x}'}$$

For a non-relativistic system,

$$v \ll c$$

$$\bar{\lambda} \gg d$$



and  $T_{kl} \neq 0$  only inside the source  $|\vec{x}'| \leq d$

$$\longrightarrow \omega \frac{\vec{x}' \cdot \hat{x}}{c} \leq \frac{\omega_s d}{c} \ll 1$$

We can therefore expand

$$e^{-i\omega(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c})} = e^{-i\omega(t - \frac{r}{c})} [1 - i\frac{\omega}{c} \vec{x}' \cdot \hat{x} + O(\frac{\omega}{c})^2]$$

This is equivalent in position space to expanding

$$T_{kl}(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c}, \vec{x}') \approx T_{kl}(t - \frac{r}{c}, \vec{x}') + \frac{\vec{x}' \cdot \hat{x}}{c} \partial_t T_{kl} + O(\partial_t^2 T_{kl})$$

We define the multipoles of the stress-tensor

$$S^{ij}(t) = \int d^3x T^{ij}(t, \vec{x})$$

$$S^{ij,k}(t) = \int d^3x T^{ij}(t, \vec{x}) x^k$$

$$S^{ij,kl}(t) = \int d^3x T^{ij}(t, \vec{x}) x^k x^l$$

We get

$$h_{ij}^{TT}(t, x) = \frac{1}{r} \frac{4G}{c} \Lambda_{ij,kl}(\hat{x}) [S^{kl} + \frac{1}{c} \hat{x}_m \dot{S}^{kl,m} + \frac{1}{2c^2} \hat{x}_m \hat{x}_p \ddot{S}^{kl,mp} + \dots]_{ret}$$

evaluated at  $t-r/c$

$$\frac{\omega_s d}{c} \sim \frac{v}{c}$$

$$O(\frac{v}{c})^2$$

Therefore, weak sources with low velocity emit gravitational radiation that is essentially determined by the lowest multipole moments

We do not need to know all the structure of the source

We only need to know its lowest multipole moments

$$h_{ij}^{TT}(t, x) = \frac{1}{r} \frac{4G}{c} \Lambda_{ij,kl}(\hat{x}) [S^{kl} + \frac{1}{c} \hat{x}_m \dot{S}^{kl,m} + \frac{1}{2c^2} \hat{x}_m \hat{x}_p \ddot{S}^{kl,mp} + \dots]_{ret}$$
$$\frac{\omega_s d}{c} \sim \frac{v}{c} \quad O\left(\frac{v}{c}\right)^2$$

In terms of GW power:

2.5PN

3.5PN

[will be justified later]

Two sets of multipoles have a physical interpretation:

Mass multipoles:

$$M = \frac{1}{c^2} \int d^3x T^{00}(t, x)$$

$$M^i = \frac{1}{c^2} \int d^3x T^{00}(t, x) x^i$$

...

Momentum multipoles:

$$P^i = \frac{1}{c} \int d^3x T^{0i}(t, x)$$

$$P^{i,j} = \frac{1}{c} \int d^3x T^{0i}(t, x) x^j$$

...

The multipoles of  $T^{ij}$  are related to time derivatives of these multipoles by the conservation of the stress-tensor

## Exercise

Obtain the time derivative of the lowest mass and momentum multipoles

$$M^L = \frac{1}{c^2} \int d^3x T^{00} x^L \quad P^{i,L} = \frac{1}{c} \int d^3x T^{0i} x^L$$

(i)  $\dot{M} = 0$

(ii)  $\dot{M}^i = P^i$

(iii)  $\dot{M}^{ij} = P^{i,j} + P^{j,i}$

(iv)  $\dot{P}^i = 0$

(v)  $\dot{P}^{i,j} = S^{ij}$

(vi)  $\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}$

using  $\partial_\mu T^{\mu\nu} = 0$ .

## Solution

$$(i) \quad c\dot{M} = \int d^3x \partial_0 T^{00} = - \int d^3x \partial_i T^{0i} = - \int dS_i T^{0i} = 0$$

for a volume larger than the source.

In linear theory, back reaction of GW are absent, and the mass of matter is conserved.

$$\begin{aligned} (ii) \quad c\dot{M}^i &= \int d^3x x^i \partial_0 T^{00} = - \int d^3x x^i \partial_j T^{0j} \\ &= \int d^3x \partial_j x^i T^{0j} = \int d^3x T^{0i} = c P^i \end{aligned}$$

## Solution

(iii)  $\dot{M}^{ij} = P^{i,j} + P^{j,i}$

$$\begin{aligned}\dot{M}_{ij} &= \frac{1}{c} \int d^3x \partial_0 T^{00} x^i x^j \\ &= -\frac{1}{c} \int d^3x \partial_k T^{0k} x^i x^j \\ &= \frac{2}{c} \int d^3x T^{0k} \partial_k x^{(i} x^{j)} \\ &= P^{i,j} + P^{j,i}\end{aligned}$$

(iv)  $\dot{P}^i = 0$  This is the conservation of momentum

(v)  $\dot{P}^{i,j} = S^{ij}$   $[ij] \rightarrow$  conservation of angular momentum

(vi)  $\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}$

We can combine these identities to express  $S^{ij}$ ,  $S^{ij,k}$  in terms of the mass and momentum multipole moments !

$$\dot{M}^{ij} = P^{i,j} + P^{j,i}$$

$$\dot{P}^{i,j} = S^{ij}$$



$$S^{ij} = \frac{1}{2} \ddot{M}^{ij}$$

The leading term of

$$h_{ij}^{TT}(t, x) = \frac{1}{r} \frac{4G}{c} \Lambda_{ij,kl}(\hat{x}) [S^{kl} + \frac{1}{c} \hat{x}_m \dot{S}^{kl,m} + \frac{1}{2c^2} \hat{x}_m \hat{x}_p \ddot{S}^{kl,mp} + \dots]_{ret}$$

becomes

$$h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl}(\vec{x}) \ddot{M}^{kl}\left(t - \frac{r}{c}\right)$$

The acceleration of mass quadrupole sources gravitational waves.

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

$$P_{ij} \equiv \delta_{ij} - n_i n_j$$

# Radiated energy

$$\begin{aligned}\frac{dE}{dtd\Omega} &= \frac{c^3 r^2}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle & h_{ij}^{TT}(t, \vec{x}) &= \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl}(\vec{x}) \ddot{M}^{kl}(t - \frac{r}{c}) \\ &= \frac{G}{8\pi c^5} \Lambda_{ij,kl}(\hat{n}) \langle \ddot{\ddot{M}}_{ij} \ddot{\ddot{M}}_{kl} \rangle_{ret} \\ &= \frac{G}{8\pi c^5} \Lambda_{ij,kl}(\hat{n}) \langle \ddot{\ddot{Q}}_{ij} \ddot{\ddot{Q}}_{kl} \rangle_{ret} & Q_{ij} - M_{ij} - \frac{1}{3} \delta_{ij} M_{kk}\end{aligned}$$

Angular integral can be done

$$\int d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15} (11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

Exercise  
Later on

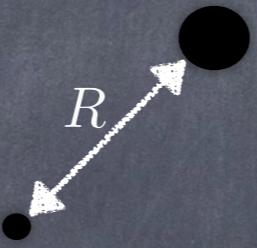
→  $\frac{dE}{dt} = \frac{G}{5c^5} \langle \ddot{\ddot{Q}}_{ij}(t - \frac{r}{c}) \ddot{\ddot{Q}}_{ij}(t - \frac{r}{c}) \rangle$

That is the Einstein quadrupole formula

By conservation of energy,

$$\frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

For a binary source,



$$E_{0PN}^{source} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{Gm_1m_2}{R}$$

$$E_{0PN}^{source} = \frac{1}{2}mv_{c.m.}^2 + \frac{1}{2}\mu\dot{R}^2 - \frac{Gm\mu}{R}$$

$$E_{0PN}^{source} = \frac{1}{2}mv_{c.m.}^2 - \frac{Gm\mu}{2R}$$

Effective one-body reduction

$$\text{Reduced mass } \mu = \frac{m_1m_2}{m_1 + m_2}$$

$$\text{Total mass } m = m_1 + m_2$$

$$\dot{R}^2 = \frac{Gm}{R}$$

Kepler Law

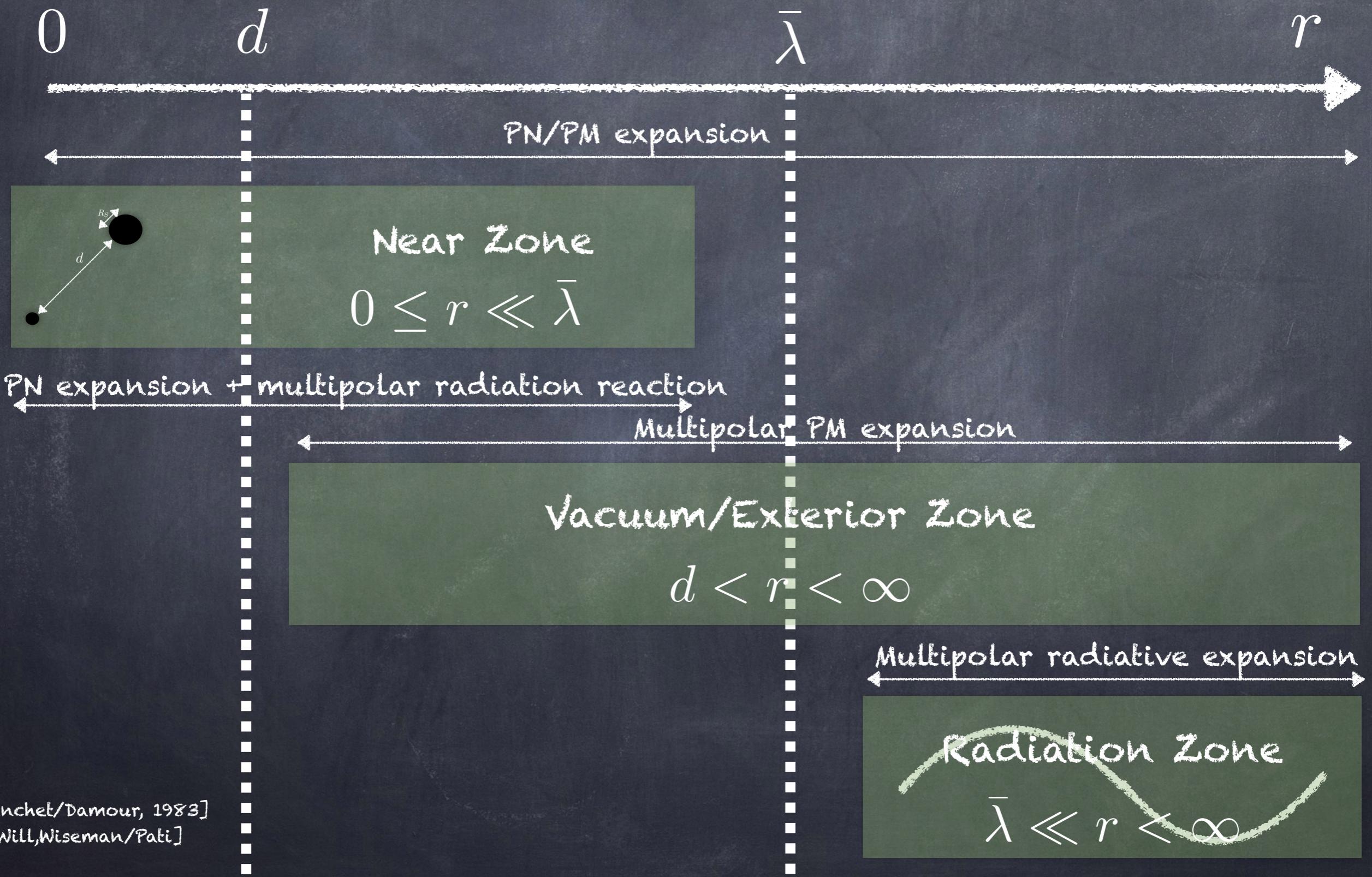
$$\frac{dE^{source}}{dt} = \frac{Gm\mu}{2R^2}\dot{R}$$

$$\frac{dE}{dt} = \frac{G}{5c^5} \langle \ddot{\vec{Q}}_{ij}(t - \frac{r}{c}) \ddot{\vec{Q}}_{ij}(t - \frac{r}{c}) \rangle$$

$$\dot{R} \sim c^{-5}$$

The gravitational wave emission brings as first contribution a 2.5PN correction to the motion. It is called the radiation reaction.

# The 3 zones of the Post-Newtonian/ Post-Minkowskian formalism



$$h_{ij}^{TT}(t, x) = \frac{1}{r} \frac{4G}{c} \Lambda_{ij,kl}(\hat{x}) [S^{kl} + \frac{1}{c} \hat{x}_m \dot{S}^{kl,m} + \frac{1}{2c^2} \hat{x}_m \hat{x}_p \ddot{S}^{kl,mp} + \dots]_{ret}$$

2.5PN

3.5PN

$$\begin{aligned} \frac{dv_1}{dt} = & -\frac{Gm_2}{r_{12}^2} \mathbf{n}_{12} \\ & + \underbrace{\frac{1}{c^2} \left\{ \left[ \frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \dots \right] \mathbf{n}_{12} + \dots \right\}}_{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} \\ & + \underbrace{\frac{1}{c^4} [\dots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5} [\dots]}_{\text{2.5PN radiation reaction}} + \underbrace{\frac{1}{c^6} [\dots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7} [\dots]}_{\text{3.5PN radiation reaction}} + \underbrace{\frac{1}{c^8} [\dots]}_{\text{4PN conservative \& radiation tail}} + \mathcal{O}\left(\frac{1}{c^9}\right) \end{aligned}$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc]	Fokker Lagrangian
	[Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019]	Effective field theory

## Exercise

Prove

$$\int d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15} (11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

where

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \quad P_{ij} \equiv \delta_{ij} - n_i n_j$$

$$n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

(i) Prove  $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin |\vec{k}|}{|\vec{k}|}$

(ii) Prove  $\int_S d\Omega n_{i_1} \cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}| \rightarrow 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1} \cdots k^{i_\ell}} \left( \frac{\sin |\vec{k}|}{|\vec{k}|} \right)$

(iii) Prove  $\int_S d\Omega n_{i_1} \cdots n_{i_\ell} = \frac{4\pi}{\ell+1} \delta_{(i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{\ell-1} i_\ell)}$  for  $\ell$  even

(iv) Prove  $\int_S d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15} (11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$

## Solution

(i) Prove  $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin |\vec{k}|}{|\vec{k}|}$   $n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$

By SO(3) symmetry, we can assume that  $\vec{k}$  points in the z direction

$$\begin{aligned} \int_S d\theta d\phi \sin \theta e^{-ik_z \cos \theta} &= -2\pi \int_1^{-1} d\cos \theta e^{-ik_z \cos \theta} \\ &= \frac{2\pi}{ik_z} (e^{+ik_z} - e^{-ik_z}) = \frac{4\pi \sin k_z}{k_z} \end{aligned}$$

Since the right-hand side is SO(3) invariant (scalar), it is a function of the norm of  $\vec{k}$ .

## Solution

(i) Prove  $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin |\vec{k}|}{|\vec{k}|}$   $n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$

(ii) Prove  $\int_S d\Omega n_{i_1} \cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}| \rightarrow 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1} \cdots k^{i_\ell}} \left( \frac{\sin |\vec{k}|}{|\vec{k}|} \right)$  trivial

## Solution

(i) Prove  $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin |\vec{k}|}{|\vec{k}|}$   $n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$

(ii) Prove  $\int_S d\Omega n_{i_1} \cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}| \rightarrow 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1} \cdots k^{i_\ell}} \left( \frac{\sin |\vec{k}|}{|\vec{k}|} \right)$  trivial

(iii) Prove  $\int_S d\Omega n_{i_1} \cdots n_{i_\ell} = \frac{4\pi}{\ell + 1} \delta_{(i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{\ell-1} i_\ell)}$

Use Taylor series

$$\frac{\sin |\vec{k}|}{|\vec{k}|} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} |\vec{k}|^{2n}$$

$$i^\ell = (-1)^n$$

Only one term survives:  $2n = \ell \rightarrow$

$$\frac{\ell!}{(2n+1)!} = \frac{1}{2n+1}$$

## Solution

(i) Prove  $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin |\vec{k}|}{|\vec{k}|}$   $n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$

(ii) Prove  $\int_S d\Omega n_{i_1} \cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}| \rightarrow 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1} \cdots k^{i_\ell}} \left( \frac{\sin |\vec{k}|}{|\vec{k}|} \right)$  trivial

(iii) Prove  $\int_S d\Omega n_{i_1} \cdots n_{i_\ell} = \frac{4\pi}{\ell + 1} \delta_{(i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{\ell-1} i_\ell)}$

Use Taylor series

$$\frac{\sin |\vec{k}|}{|\vec{k}|} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} |\vec{k}|^{2n}$$

$$i^\ell = (-1)^n$$

Only one term survives:  $2n = \ell \rightarrow$

$$\frac{\ell!}{(2n+1)!} = \frac{1}{2n+1}$$

(iv) Prove  $\int_S d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15} (11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$   $\frac{4!}{2 \cdot 2 \cdot 2} = 3$

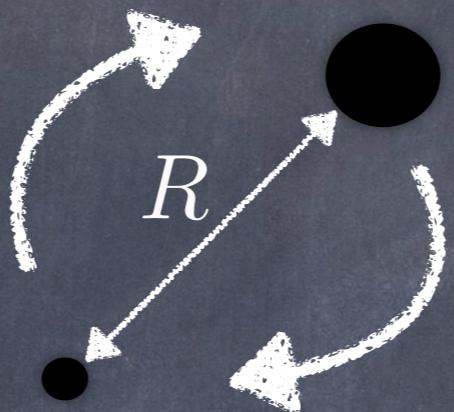
Use  $\int_S d\Omega 1 = 4\pi, \quad \int_S d\Omega n_i n_j = \frac{4\pi}{3} \delta_{ij}, \quad \int_S d\Omega n_i n_j n_k n_l = \frac{4\pi}{15} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})$

## 1.4. Quasi-circular inspiral of compact binaries

- Neglect conservative PN corrections  
(Keplerian motion + radiation reaction)

- Assume quasi-circularity

Kepler Law  $\omega_s^2 = \frac{Gm}{R^3}$



Total Mass

$$m = m_1 + m_2$$

Reduced Mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Law of motion given by the flux-balance law of energy:

$$\frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

$$\frac{dE^{source}}{dt} = \frac{Gm\mu}{2R^2} \dot{R}$$

$$\frac{dE}{dt} = \frac{G}{5c^5} \langle \ddot{Q}_{ij}(t - \frac{r}{c}) \dot{Q}_{ij}(t - \frac{r}{c}) \rangle$$

## Exercise

Obtain the quadrupole radiation from a mass in a circular orbit

(i) Starting from the relativistic expression

$$T^{\mu\nu} = \sum_{A=1,2} \int d\tau_A m_A \frac{dx_A^\mu}{d\tau} \frac{dx_A^\nu}{d\tau} \delta^{(4)}(x - x_A(\tau_A))$$

prove that at Newtonian order:

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

(ii) Deduce the quadrupole formula

$$Q^{ij}(t) = \left[ \frac{1}{c^2} \int d^3x T^{00}(t, \vec{x}) x^i x^j \right]^{STF} = \mu \left( x_0^i(t) x_0^j(t) - \frac{1}{3} r_0^2(t) \delta^{ij} \right)$$

where  $\vec{x}_0 \equiv \vec{x}_1 - \vec{x}_2$

## Solution

(i)

$$T^{\mu\nu} = \sum_{A=1,2} \int d\tau_A m_A \frac{dx_A^\mu}{d\tau_A} \frac{dx_A^\nu}{d\tau_A} \delta^{(4)}(x - x_A(\tau_A))$$

$$c^2 d\tau_A^2 = -\eta_{\mu\nu} dx_A^\mu dx_A^\nu = c^2 (1 - \frac{v_A^2}{c^2}) dt^2 = c^2 \gamma_A^{-2} dt^2 \quad \longrightarrow \quad d\tau_A = \frac{dt}{\gamma_A}$$

$$T^{\mu\nu} = \sum_{A=1,2} \gamma_A m_A \frac{dx_A^\mu}{dt} \frac{dx_A^\nu}{dt} \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} \gamma_A m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

**Newtonian Limit:**

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

(ii)

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

$$M^{ij} = \frac{1}{c^2} \int d^3x T^{00}(t, \vec{x}) x^i x^j$$

$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m x_{c.m.}^i x_{c.m.}^j + \mu x_0^i x_0^j$$



$$\vec{x}_0 \equiv \vec{x}_1 - \vec{x}_2$$

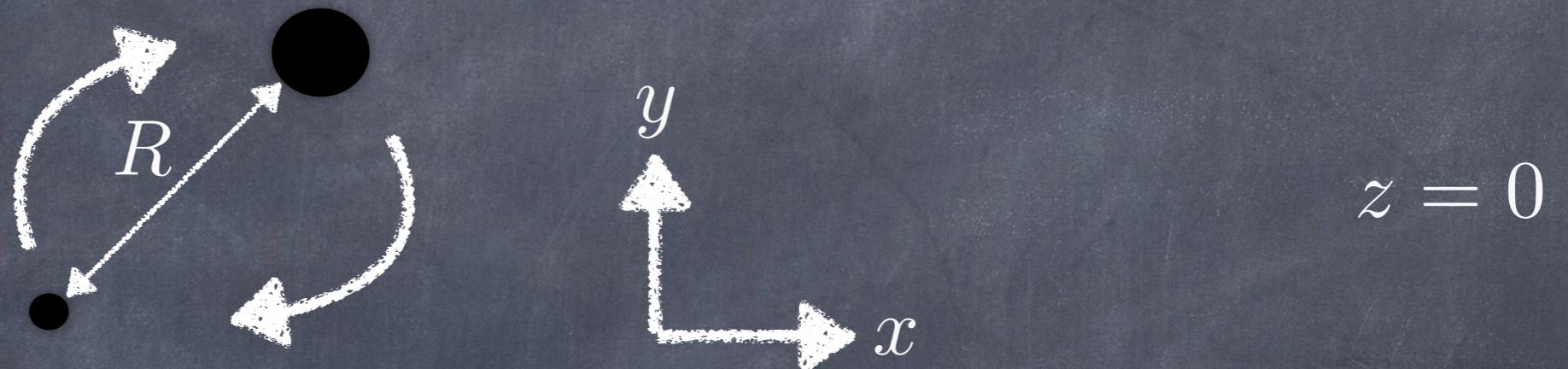
$$\vec{x}_{c.m.} \equiv \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} = 0$$

$$Q^{ij}(t) = \left[ \frac{1}{c^2} \int d^3x T^{00}(t, \vec{x}) x^i x^j \right]^{STF}$$

$$Q^{ij}(t) = \mu \left( x_0^i(t) x_0^j(t) - \frac{1}{3} r_0^2(t) \delta^{ij} \right)$$

# Explicit expression

Assume orbital motion is exactly circular (not elliptic)



$$z = 0$$

$$\begin{cases} x_0(t) = R \cos(\omega_s t + \frac{\pi}{2}) \\ y_0(t) = R \sin(\omega_s t + \frac{\pi}{2}) \\ z_0(t) = 0 \end{cases}$$

$$M^{11} = \mu R^2 \cos^2(\omega_s t + \frac{\pi}{2})$$

$$M^{ij} = \mu x_0^i(t) x_0^j(t)$$

$$M^{12} = \mu R^2 \sin(\omega_s t + \frac{\pi}{2}) \cos(\omega_s t + \frac{\pi}{2})$$

$$M^{22} = \mu R^2 \sin^2(\omega_s t + \frac{\pi}{2})$$

# Metric perturbation

$$h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl}(\vec{x}) \ddot{M}^{kl}\left(t - \frac{r}{c}\right)$$

For a generic direction  $\hat{n} = (\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta)$   
we find

$$\begin{aligned} h_+(t, \theta, \phi) &= \frac{1}{r} \frac{G}{c^4} [\ddot{M}_{11}(\cos^2 \phi - \sin^2 \phi \cos^2 \theta) \\ &\quad + \ddot{M}_{22}(\sin^2 \phi - \cos^2 \phi \cos^2 \theta) \\ &\quad - \ddot{M}_{33} \sin^2 \theta \\ &\quad - \ddot{M}_{12} \sin 2\phi(1 + \cos^2 \theta) \\ &\quad + \ddot{M}_{13} \sin \phi \sin 2\theta \\ &\quad + \ddot{M}_{23} \cos \phi \sin 2\theta] \\ h_\times(t, \theta, \phi) &= \frac{1}{r} \frac{G}{c^4} [(\ddot{M}_{11} - \ddot{M}_{22}) \sin 2\phi \cos \theta \\ &\quad + 2\ddot{M}_{12} \cos 2\phi \cos \theta \\ &\quad - 2\ddot{M}_{13} \cos \phi \sin \theta \\ &\quad + 2\ddot{M}_{23} \sin \phi \sin \theta] \end{aligned}$$

$$h_+(t, \theta, \phi) = -\frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \left( \frac{1 + \cos^2 \theta}{2} \right) \cos(2\omega_s t_{ret} + 2\phi)$$

$$h_\times(t, \theta, \phi) = \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \cos \theta \sin(2\omega_s t_{ret} + 2\phi)$$

Quadrupole radiation is at twice the frequency of the source.

$$\omega_{gw} \sim 2\omega_s \quad [\text{as announced earlier}]$$

Helicoidal structure

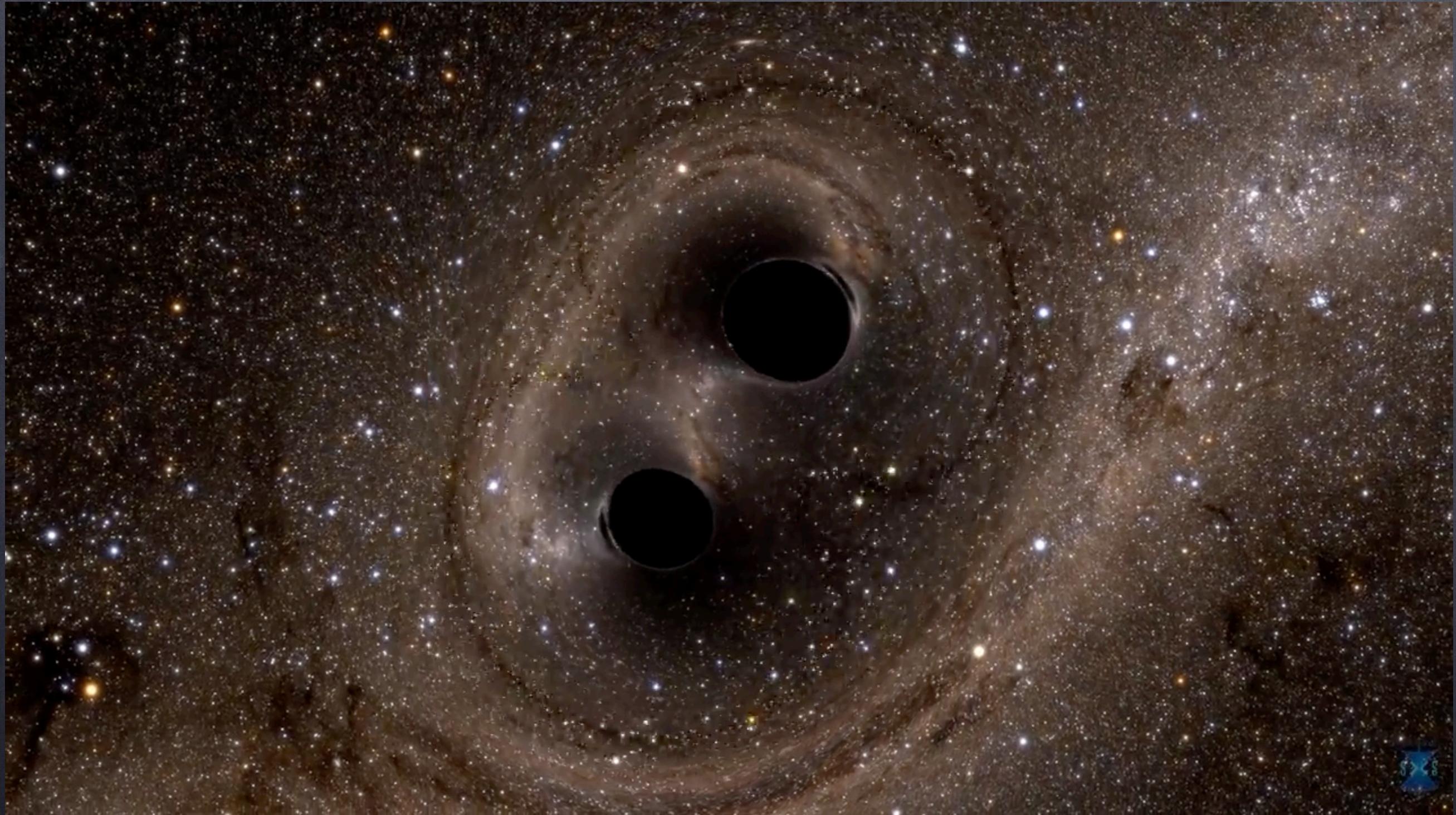
Performing a rotation is equivalent to shifting time

The angle  $\theta$  is the angle between the normal to the orbit and the line-of-sight

The distance  $r$  is a constant

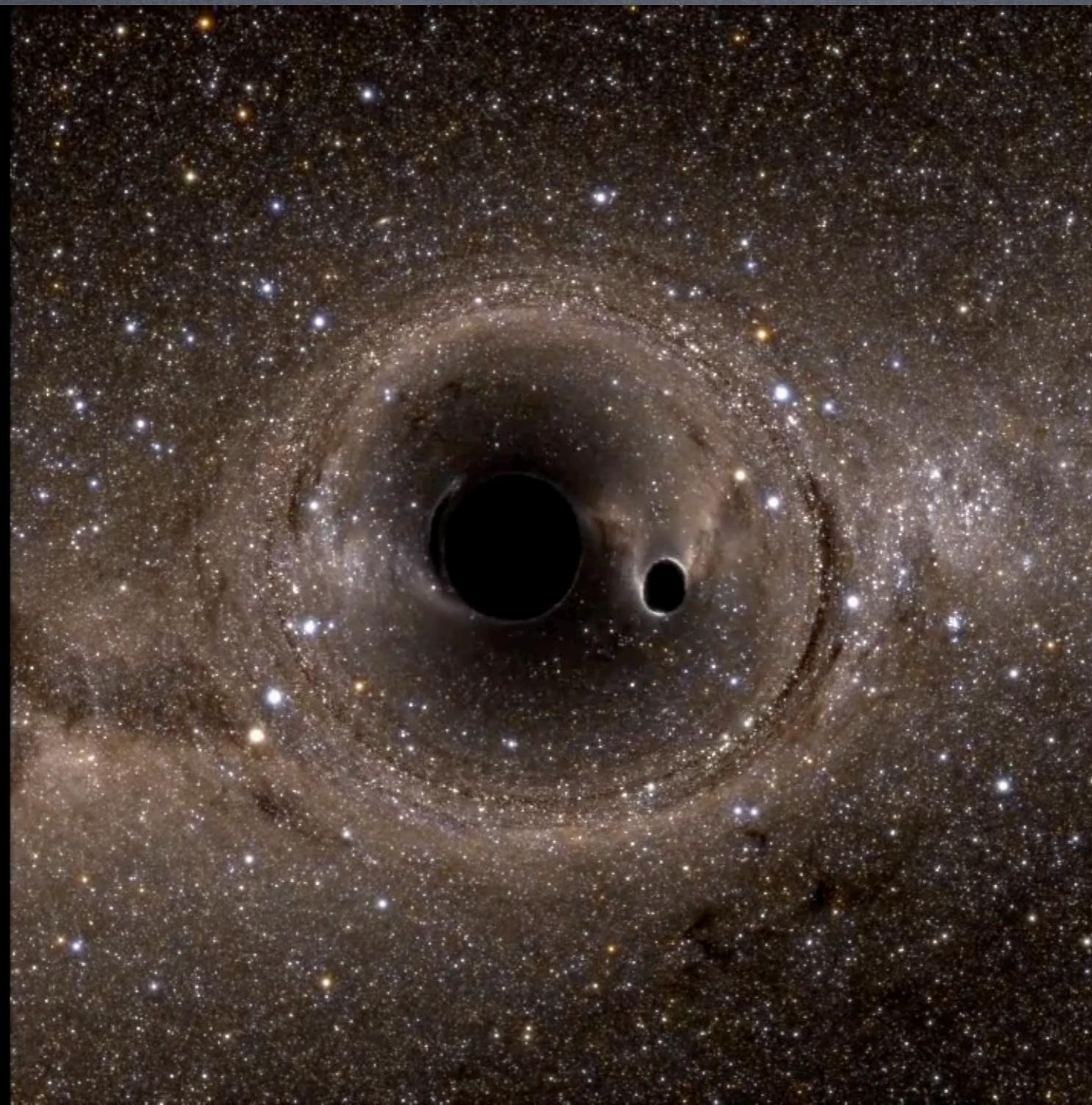
Face-on

$\theta = 0$



Edge-on

$$\theta = \frac{\pi}{2}$$



SXS Lensing

Compute the angular distribution of radiated power in the quadrupole approximation

$$\frac{dE}{dtd\Omega} = \frac{r^2 c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

using

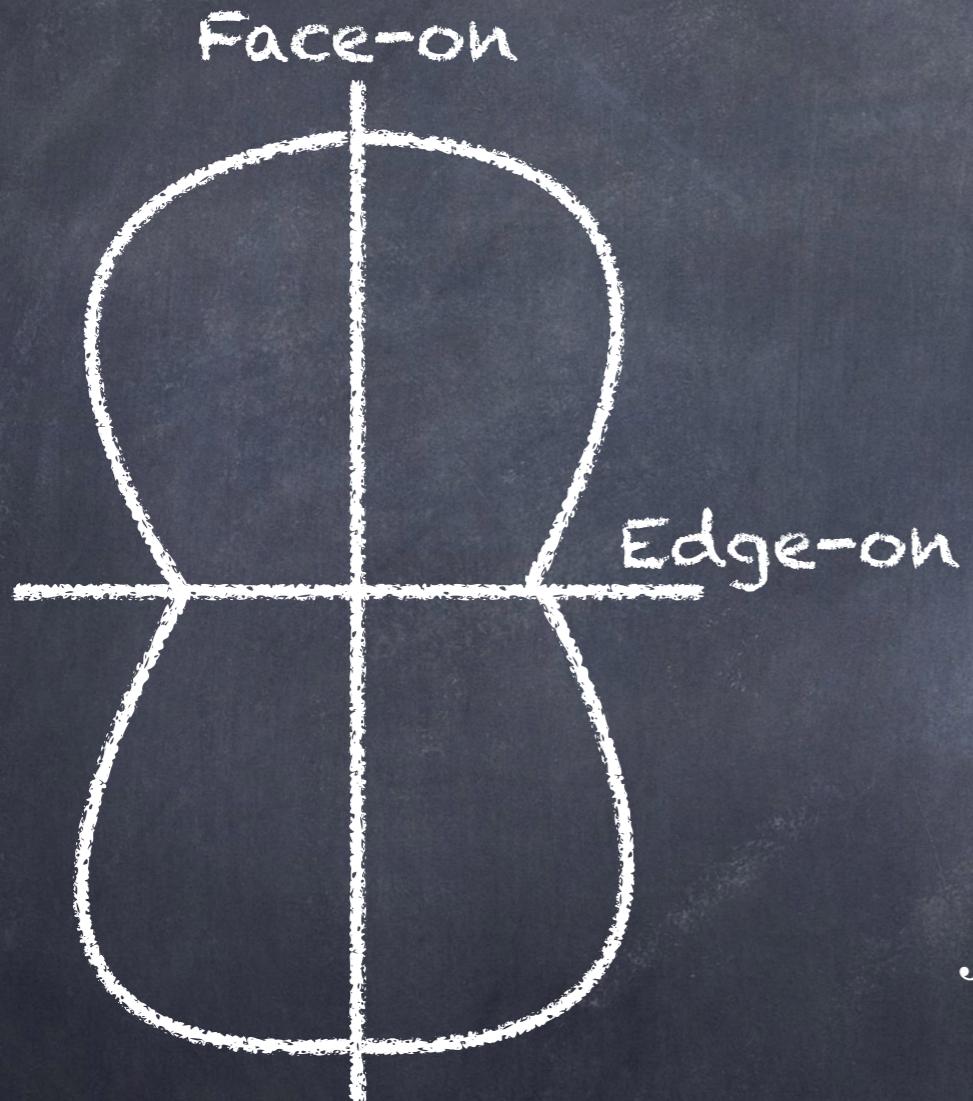
$$h_+(t, \theta, \phi) = \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \left( \frac{1 + \cos^2 \theta}{2} \right) \cos(2\omega_s t_{ret} + 2\phi)$$

$$h_\times(t, \theta, \phi) = \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \cos \theta \sin(2\omega_s t_{ret} + 2\phi)$$

Use

$$\langle \cos^2(2\omega_s t_{ret} + \phi) \rangle = \langle \sin^2(2\omega_s t_{ret} + \phi) \rangle = \frac{1}{2}$$

$$\frac{dE}{dtd\Omega} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$



$$g(\theta) = \left( \frac{1 + \cos^2 \theta}{2} \right)^2 + \cos^2 \theta$$

$$\int \frac{d\Omega}{4\pi} g(\theta) = \frac{4}{5}$$

$$P_{rad} = \frac{dE}{dt} = \frac{32}{5} \frac{G\mu^2}{c^5} R^4 \omega_s^6 = \frac{32}{5} \frac{G\mu^2}{R^2} \left( \frac{v}{c} \right)^6$$

# Inspiral motion to coalescence as a consequence of energy flux-balance

Emission of GW costs energy, taken from the system's total energy

$$\frac{dE^{source}}{dt} = \frac{Gm\mu}{2R^2} \dot{R}$$

$$\frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

- R decreases with time
- $\omega_s^2 = \frac{Gm}{R^3}$  increases with time
- power radiated in GW increases
- Runaway process leading to coalescence

Exercise

The motion is quasi-circular as long as  $\dot{R} \ll v$

Prove that it is equivalent to  $\frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1$

Solution

$$R = \left( \frac{Gm}{\omega_s^2} \right)^{1/3}$$

Kepler's Law

$$\begin{aligned}\dot{R} &= -\frac{2}{3} R \frac{\dot{\omega}_s}{\omega_s} \\ &= -\frac{2}{3} (\omega_s R) \frac{\dot{\omega}_s}{\omega_s^2} \\ &= -\frac{2}{3} v \frac{\dot{\omega}_s}{\omega_s^2}\end{aligned}$$

$$\dot{R} \ll v \Leftrightarrow \frac{\dot{\omega}_s}{\omega_s^2} \ll 1 \Leftrightarrow \frac{\dot{\omega}_{gw}}{\omega_{gw}^2} \ll 1 \Leftrightarrow \frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1$$

## Exercise

Rewrite

$$h_+(t, \theta, \phi) = \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \left( \frac{1 + \cos^2 \theta}{2} \right) \cos(2\omega_s t_{ret} + 2\phi)$$

$$h_\times(t, \theta, \phi) = \frac{1}{r} \frac{4G\mu\omega_s^2 R^2}{c^4} \cos \theta \sin(2\omega_s t_{ret} + 2\phi)$$

using

$$R = \left( \frac{Gm}{\omega_s^2} \right)^{1/3} \quad \omega_s = \frac{1}{2} \omega_{gw} = \frac{1}{2} 2\pi f_{gw} = \pi f_{gw}$$

and the chirp mass

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

Keep  $M_c$ ,  $m$  and  $f_{gw}$ . What do you notice?

## Solution

$$h_+(t) = \frac{4}{r} \left( \frac{GM_c}{c^2} \right)^{5/3} \left( \frac{\pi f_{gw}}{c} \right)^{2/3} \frac{1 + \cos^2 \theta}{2} \cos(2\pi f_{gw} t_{ret} + 2\phi)$$

$$h_\times(t) = \frac{4}{r} \left( \frac{GM_c}{c^2} \right)^{5/3} \left( \frac{\pi f_{gw}}{c} \right)^{2/3} \cos \theta \sin(2\pi f_{gw} t_{ret} + 2\phi)$$

All dependence in the mass is through the chirp mass at this lowest PN order!

Power radiated:

$$\frac{dP}{d\Omega} = \frac{2}{\pi} \frac{c^5}{G} \left( \frac{GM_c \omega_{gw}}{2c^3} \right)^{10/3} g(\theta)$$

$$P = \frac{32}{5} \frac{c^5}{G} \left( \frac{GM_c \omega_{gw}}{2c^3} \right)^{10/3}$$

## Exercise

(i) Prove

$$E^{source} = -\frac{G\mu m}{2R} = -\left(\frac{G^2 M_c^5 \omega_{gw}^2}{32}\right)^{1/3}$$

using

$$\omega_{gw} = 2\omega_s \quad R = \left(\frac{Gm}{\omega_s^2}\right)^{1/3} \quad M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

(ii) Derive the evolution of the GW frequency  $f_{gw} = \frac{\omega_{gw}}{2\pi}$

using the energy flux-balance law with GW power

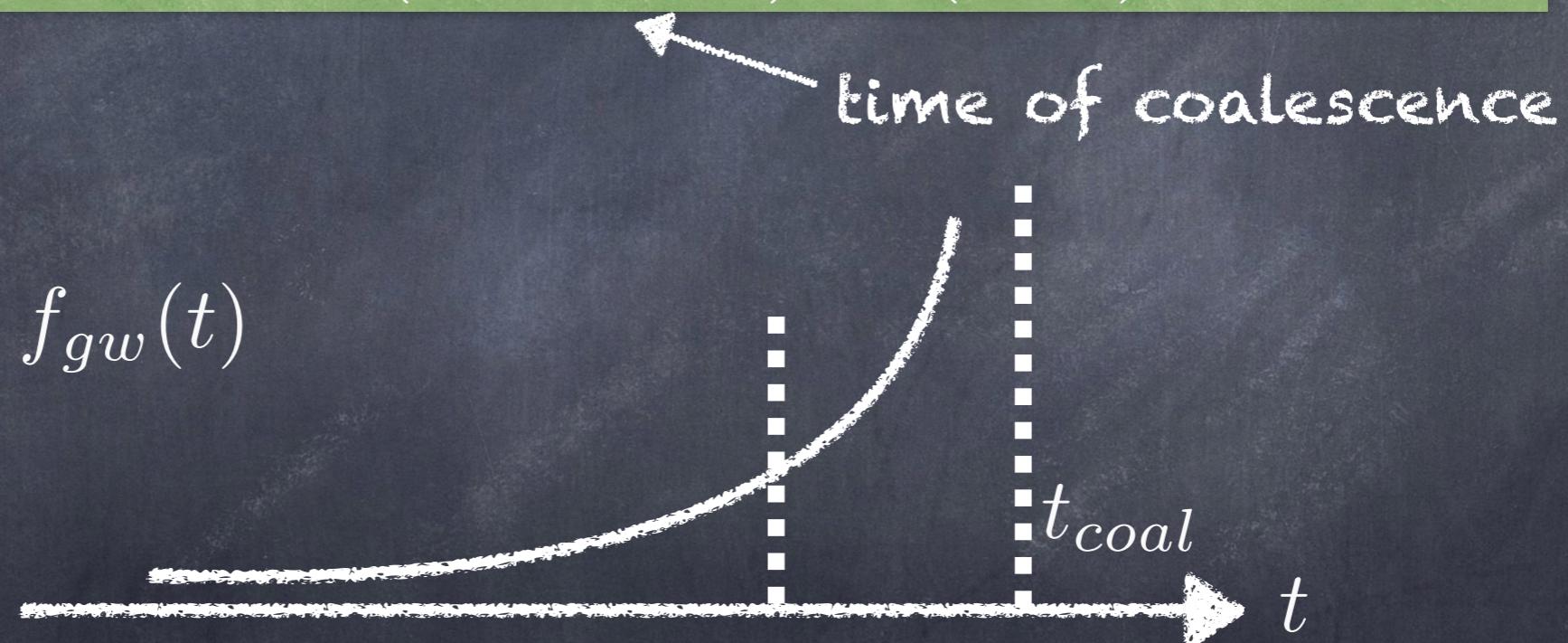
$$P = \frac{32}{5} \frac{c^5}{G} \left(\frac{GM_c \omega_{gw}}{2c^3}\right)^{10/3}$$

## Solution

$$\dot{f}_{gw} = \frac{96}{5} \pi^{8/3} \left( \frac{GM_c}{c^3} \right)^{5/3} f_{gw}^{11/3}$$

It integrates to

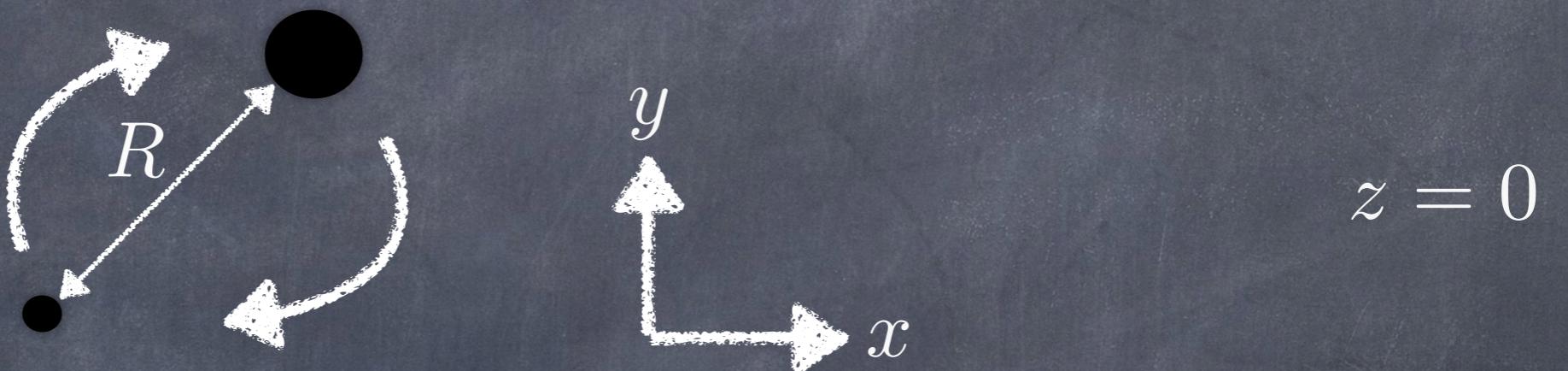
$$f_{gw}(t) = \frac{1}{\pi} \left( \frac{5}{256} \frac{1}{t_{coal} - t} \right)^{3/8} \left( \frac{GM_c}{c^3} \right)^{-5/8}$$



Validity of the quasi-circularity hypothesis:

$$\frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1 \Leftrightarrow \left( \frac{96}{5} \pi^{8/3} \right)^{3/5} \frac{GM_c}{c^3} f_{gw} \ll 1$$

To get the waveform, we need the orbital motion with 2.5PN radiation-reaction  
 (Matching between Near Zone and Exterior zone)



$$\left\{ \begin{array}{l} x_0(t) = R \cos(\omega_s t + \frac{\pi}{2}) \\ y_0(t) = R \sin(\omega_s t + \frac{\pi}{2}) \\ z_0(t) = 0 \end{array} \right. \longrightarrow \begin{array}{l} R(t) \cos \frac{\Phi(t)}{2} \\ R(t) \sin \frac{\Phi(t)}{2} \end{array}$$

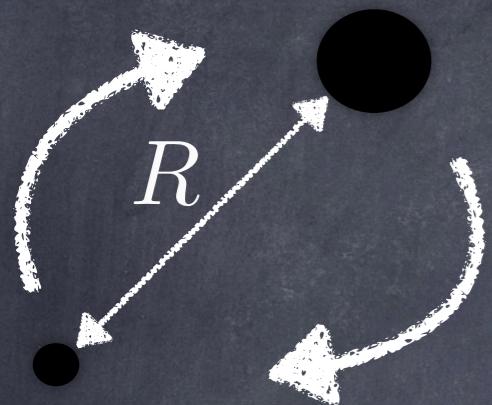
where

$$\Phi(t) = \int_{t_0}^t dt' \omega_{gw}(t') = 2 \int_{t_0}^t dt' \omega_s(t')$$

$$\Phi(t) = \Phi_0 - 2 \left( \frac{5GM_c}{c^3} \right)^{-5/8} (t_{coal} - t)^{5/8}$$

→ Quadrupole moment/Waveform gets corrections.

# Quadrupole moment with radiation-reaction



$$M^{ij} = \mu x_0^i(t) x_0^j(t)$$

$$M^{11} = \mu R^2 \cos^2(\omega_s t + \frac{\pi}{2})$$

$$M^{12} = \mu R^2 \sin(\omega_s t + \frac{\pi}{2}) \cos(\omega_s t + \frac{\pi}{2})$$

$$M^{22} = \mu R^2 \sin^2(\omega_s t + \frac{\pi}{2})$$

Replace

$$R \rightarrow R(t)$$

$$\omega_s t + \frac{\pi}{2} \rightarrow \Phi(t)/2$$

However,  $\dot{R}$  is negligible as long as the orbit is quasi-circular. Therefore, we can ignore these terms.

Therefore, the waveform  $h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl}(\vec{x}) \ddot{M}^{kl}(t - \frac{r}{c})$

is simply obtained by making the replacement  $\omega_s t + \frac{\pi}{2} \rightarrow \Phi(t)/2$

# Final OPN+2.5PN rad-react/1PM GW waveform

$$h_+(t) = \frac{4}{r} \left( \frac{GM_c}{c^2} \right)^{5/3} \left( \frac{\pi f_{gw}(t_{ret})}{c} \right)^{2/3} \left( \frac{1 + \cos^2 \theta}{2} \right) \cos \Phi(t_{ret})$$

$$h_\times(t) = \frac{4}{r} \left( \frac{GM_c}{c^2} \right)^{5/3} \left( \frac{\pi f_{gw}(t_{ret})}{c} \right)^{2/3} (\cos \theta) \sin \Phi(t_{ret})$$

$$\Phi(t) = \Phi_0 - 2 \left( \frac{5GM_c}{c^3} \right)^{-5/8} (t_{coal} - t)^{5/8}$$

where

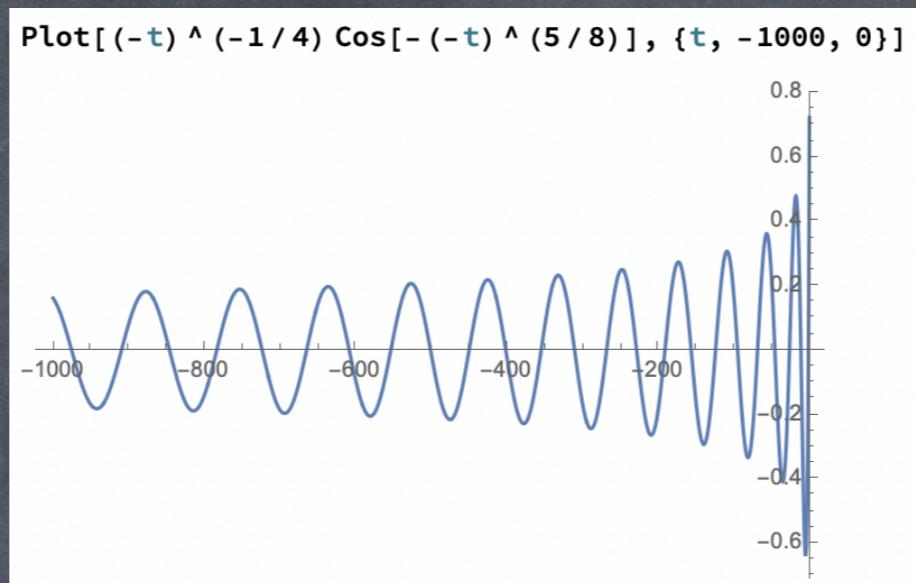
$$f_{gw}(t) = \frac{1}{\pi} \left( \frac{5}{256} \frac{1}{t_{coal} - t} \right)^{3/8} \left( \frac{GM_c}{c^3} \right)^{-5/8}$$

The retardation is a fixed time shift. Instead we express the waveform as a function of time before coalescence.

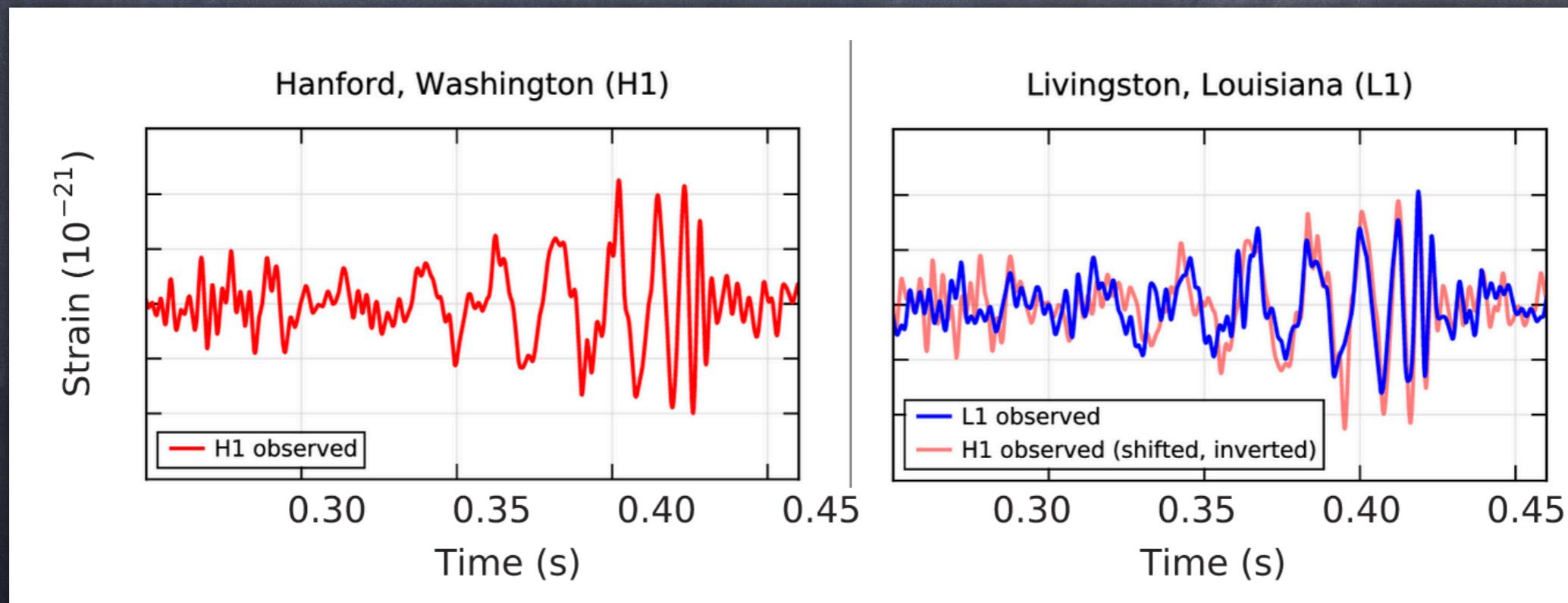
$$h_+(t) = \frac{1}{r} \left( \frac{GM_c}{c^2} \right)^{5/4} \left( \frac{5}{c(t_{coal} - t)} \right)^{1/4} \left( \frac{1 + \cos^2 \theta}{2} \right) \cos \Phi(t_{coal} - t)$$

$$h_\times(t) = \frac{1}{r} \left( \frac{GM_c}{c^2} \right)^{5/4} \left( \frac{5}{c(t_{coal} - t)} \right)^{1/4} \cos \theta \sin \Phi(t_{coal} - t)$$

# Compare OPN+2.5PN rad-react/1PM GW waveform

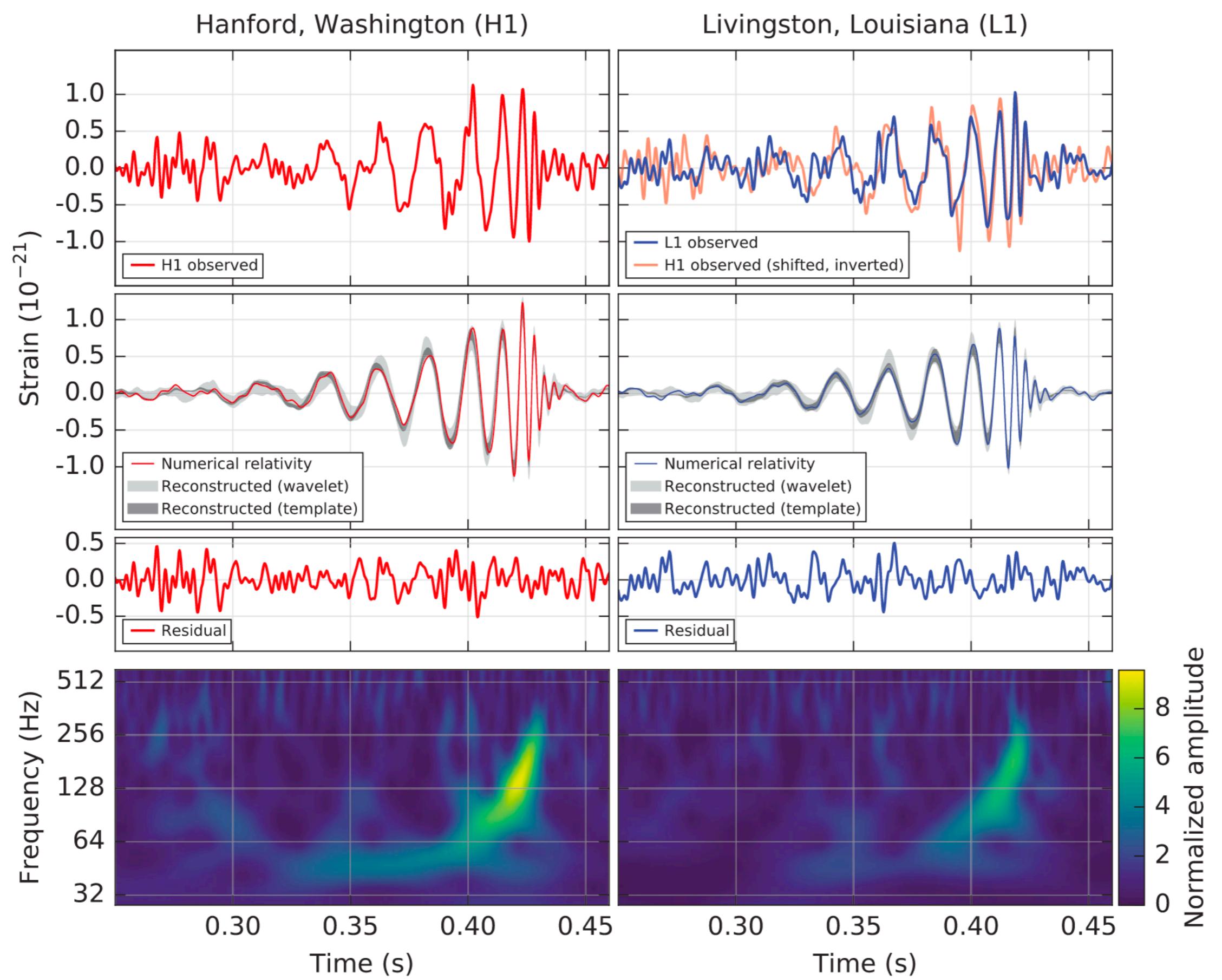


with GW150914



What do you observe?

# Actual GW template used



## Exercise

# Match with GW150914

### Estimated source parameters

Quantity	Value	Upper/Lower error estimate	Unit
Primary black hole mass	<b>36.2</b>	+5.2 -3.8	M sun
Secondary black hole mass	<b>29.1</b>	+3.7 -4.4	M sun
Final black hole mass	<b>62.3</b>	+3.7 -3.1	M sun
Final black hole spin	<b>0.68</b>	+0.05 -0.06	
Luminosity distance	<b>420</b>	+150 -180	Mpc
Source redshift, $z$	<b>0.09</b>	+0.03 -0.04	
Energy radiated	<b>3.0</b>	+0.5 -0.5	M sun

$$\frac{2GM_{\odot}}{c^2} = 3 \text{ km}$$

$$1 \text{ pc} = 3.3 \text{ light-years}$$

$$1 \text{ light-year} = 9.5 \times 10^{15} \text{ m}$$

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

(i) Obtain the GW amplitude 0.2 sec before merger

$$\mathcal{A}(t_{coal} - t) = \frac{1}{r} \left( \frac{GM_c}{c^2} \right)^{5/4} \left( \frac{5}{c(t_{coal} - t)} \right)^{1/4}$$

(ii) Obtain the GW frequency 0.2 sec before merger

$$f_{gw}(t_{coal} - t) = \frac{1}{\pi} \left( \frac{5}{256} \frac{1}{t_{coal} - t} \right)^{3/8} \left( \frac{GM_c}{c^3} \right)^{-5/8}$$

## Solution

$$(i) \quad M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \longrightarrow M_c = 28 M_\odot$$

$$\frac{2GM_\odot}{c^2} = 3 \text{ km} \longrightarrow \frac{GM_c}{c^2} = 42 \text{ km}$$

$$420 \text{ Mpc} = 1.3 \times 10^{22} \text{ km} \quad \frac{(42)^{5/4} 5^{1/4}}{1.3 \times 10^{22} (3 \times 10^5)^{1/4} (0.2)^{1/4}} = 7.8 \times 10^{-22}$$

$$\mathcal{A}(t) = 7.8 \times 10^{-22} \left( \frac{420 \text{ Mpc}}{r} \right) \left( \frac{M_c}{28 M_\odot} \right)^{5/4} \left( \frac{0.2 \text{ sec}}{t_{coal} - t} \right)^{1/4}$$

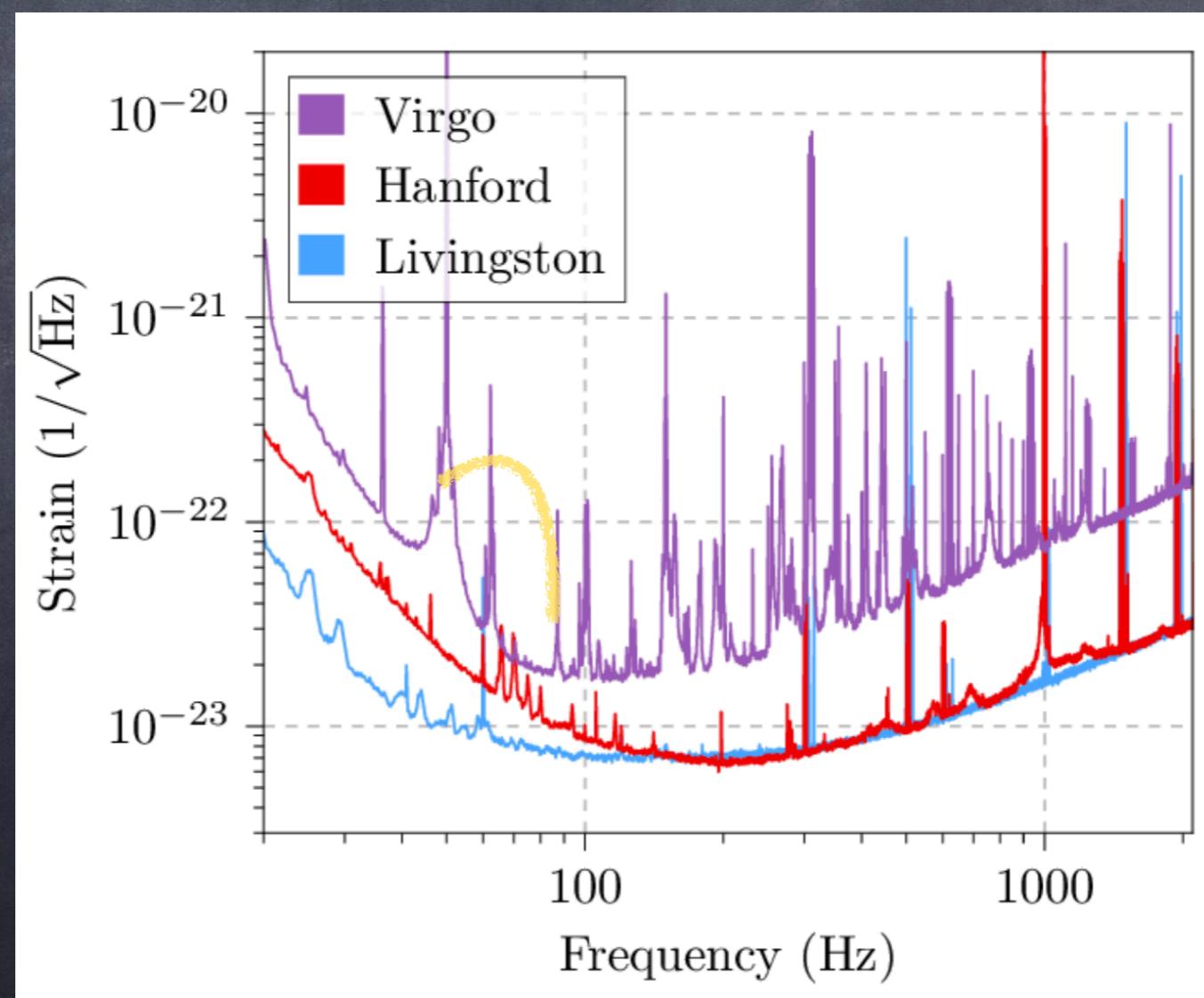
$$(ii) \quad \frac{1}{\pi} \left( \frac{5}{256} \right)^{3/8} \frac{42^{-5/8} (3 \times 10^5)^{5/8}}{0.2^{3/8}} = 34$$

$$f_{gw}(t) = 34 \text{ Hz} \left( \frac{M_c}{28 M_\odot} \right)^{-5/8} \left( \frac{0.2 \text{ sec}}{t_{coal} - t} \right)^{3/8}$$

# Model versus LIGO sensitivity

$$\mathcal{A}(t) = 7.8 \times 10^{-22} \left( \frac{420 \text{ Mpc}}{r} \right) \left( \frac{M_c}{28M_\odot} \right)^{5/4} \left( \frac{0.2 \text{ sec}}{t_{coal} - t} \right)^{1/4}$$

$$f_{gw}(t) = 34 \text{ Hz} \left( \frac{M_c}{28M_\odot} \right)^{-5/8} \left( \frac{0.2 \text{ sec}}{t_{coal} - t} \right)^{3/8}$$



# Limit of the PN/PM formalism

The PN/PM formalism leads to a perturbation series around a Keplerian orbit. After taking into account the backreaction of the central black hole, relativistic effects come in.

Schwarzschild admits an innermost stable circular orbit (ISCO)

$$r_{ISCO} = \frac{6Gm}{c^2}$$

The quasi-circularity breaks down because there are no more stable circular orbits beyond that radius. This regime is called the transition to merger. The maximal frequency where the PN/PM formalism is valid can be approximated to

$$f_{gw}^{MAX} = 2f_s = \frac{2\omega_s}{2\pi} = \frac{1}{\pi} \sqrt{\frac{Gm}{R^3}} = \frac{1}{6\sqrt{6}\pi} \frac{c^3}{Gm} \sim 34 \text{ Hz} \frac{65M_\odot}{m}$$

Note that it occurs before the end of the quasi-circularity hypothesis computed earlier:

$$f_{gw}^{MAX,QC} = \left(\frac{96}{5}\pi^{8/3}\right)^{-3/5} \frac{c^3}{GM_c} \sim 194 \text{ Hz} \frac{28M_\odot}{M_c}$$

## 1.5. Post-Newtonian corrections

# Post-Newtonian corrections to the motion

$$\begin{aligned}
 \frac{dv_1}{dt} = & -\frac{Gm_2}{r_{12}^2} \mathbf{n}_{12} \\
 & + \underbrace{\frac{1}{c^2} \left\{ \left[ \frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \dots \right] \mathbf{n}_{12} + \dots \right\}}_{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} \\
 & + \underbrace{\frac{1}{c^4} [\dots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5} [\dots]}_{\text{2.5PN radiation reaction}} + \underbrace{\frac{1}{c^6} [\dots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7} [\dots]}_{\text{3.5PN radiation reaction}} + \underbrace{\frac{1}{c^8} [\dots]}_{\text{4PN conservative \& radiation tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)
 \end{aligned}$$

2PN : [Ohta et al., 1975], [Deruelle, Damour, 1981]

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc]	Fokker Lagrangian
	[Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019]	Effective field theory

# Newtonian sources

$$\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} \int d^4x' G(x - x') T_{\mu\nu}(x')$$

$$G(x - x') = \frac{-1}{4\pi |\vec{x} - \vec{x}'|} \delta(x_{ret}^0 - x'^0_{ret})$$

For Newtonian sources:  $T_{00} \gg |T_{0i}|, |T_{ij}|$

$$\bar{h}_{00} = -4\phi(\mathbf{x}) \quad \bar{h}_{0i} = 0, \quad \bar{h}_{ij} = 0.$$

$$\phi(\mathbf{x}) = -G \int \frac{c^{-3} T_{00}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + (\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h})$$

$$g_{00} = -(1 + 2\phi), \quad g_{0i} = 0, \quad g_{ij} = (1 - 2\phi)\delta_{ij}$$

$$g_{00} = -(1 + 2\phi), \quad g_{0i} = 0, \quad g_{ij} = (1 - 2\phi)\delta_{ij}$$

$$\phi(\mathbf{x}) = -G \int \frac{c^{-3} T_{00}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

In the monopolar approximation,

$$\phi = -\frac{GM}{c^2 r} = -\frac{m}{r}$$

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 + \frac{2m}{r})(dx^2 + dy^2 + dz^2)$$

Multipolar expansion:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} \stackrel{r \geq r'}{=} \frac{4\pi}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \left(\frac{r'}{r}\right)^l Y_{lm}^*(\hat{\mathbf{x}}') Y_{lm}(\hat{\mathbf{x}})$$

$$\phi(\mathbf{x}) = -4\pi G \sum_{l,m} \frac{Q_{lm}^*}{2l+1} \frac{1}{r^{l+1}} Y_{lm}(\hat{\mathbf{x}}), \quad Q_{lm} = \int \rho(\mathbf{x}') r'^l Y_{lm}(\mathbf{x}') d^3x'$$

# The "effacement" principle or "skeletonization"

[Brillouin, Levi-Civita, Damour]

"As far as gravitational effects are concerned, the internal structure of slowly moving bodies leads to very small corrections with respect to the point particle approximation."

Assume two nearly spherically symmetric bodies of linear dimension  $L$ , of mass  $M$ , separated by a distance  $R$ .

Self-gravitational force on one body:  $\frac{GM}{L^2}$

Tidal gravitational force on one body caused by the other:  $L \frac{d}{dr} \left( \frac{GM}{r^2} \right) = \frac{GM}{R^3} L$

Tidal distortion ellipticity :  $\varepsilon \sim \frac{\text{tidal}}{\text{self}} \sim \frac{L^3}{R^3}$

Tidally induced quadrupole moment  $J_2 = \epsilon M L^2$

Force induced by the quadrupole :

$$J_2 \frac{GM}{R^3} \sim \frac{G\epsilon M^2 L^2}{R^4}$$

Force due to tidal forces

Newtonian force

$$\frac{\frac{G\epsilon M^2 L^2}{R^4}}{\frac{GM^2}{R^2}} \sim \frac{\epsilon L^2}{R^2} \sim \left(\frac{L}{R}\right)^5$$

For a compact object,  $L \sim \frac{GM}{c^2}$

Slow motion/Weak field:  $\frac{v^2}{c^2} \sim \frac{GM}{Rc^2}$



Force due to tidal forces  $\sim \left(\frac{v^2}{c^2}\right)^5$

Newtonian force

SPN effect

# Relativistic tidal deformability

$$\frac{1-g_{tt}}{2} = -\frac{m}{r} - \frac{3Q_{ij}}{2r^3} \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) + O(r^{-4}) \\ + \frac{1}{2} \mathcal{E}_{ij} x^i x^j + O(r^3),$$

Tidal deformability

$$\lambda \equiv -\frac{Q_{ij}}{\mathcal{E}_{ij}}.$$

Love number

$$\lambda = \frac{2}{3} k_2 R^5,$$

Dimensionless tidal deformability

$$\Lambda \equiv \frac{\lambda}{m^5}$$

It is 0 for black holes [due to hidden symmetries], non-zero for neutron stars.

GW depend at leading SPN order only on the combination:

$$\tilde{\Lambda} \equiv \frac{16}{3} \frac{(m_1 + 12m_2)m_1^4 \Lambda_1 + (m_2 + 12m_1)m_2^4 \Lambda_2}{(m_1 + m_2)^5},$$

[Flanagan, Hinderer, 2007]

[Favata, 2013]

Of interest to extract information on the composition of Neutron Stars.

Field equations in the PN  
approximation: the 1PN Einstein-  
Infeld-Hoffmann equations

## 1.6. Effective one-body resummation

board

# Challenges within the PN/PM formalism

[Blanchet/Damour, 1988]

