Lectures on Gravitational Waves

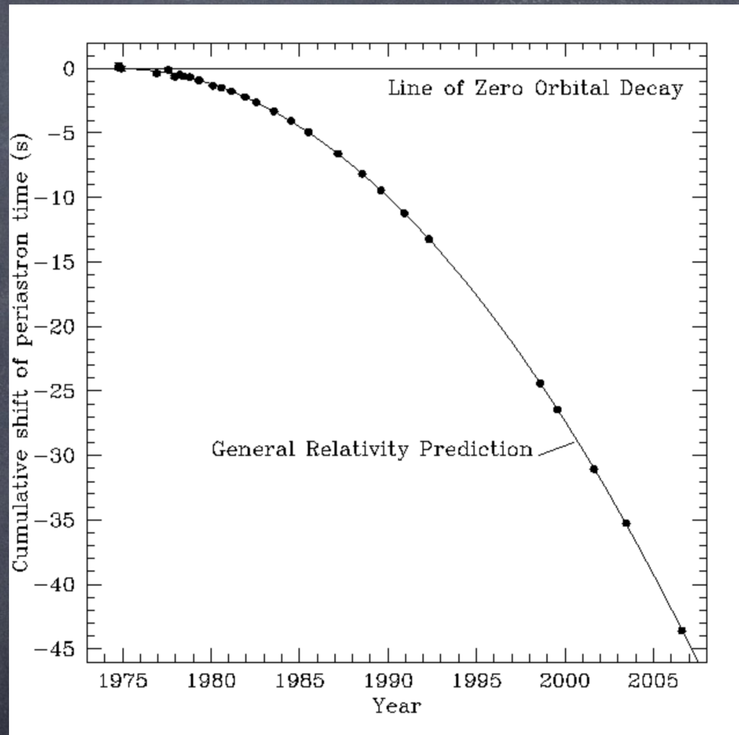
Geoffrey Compère

Solvay Doctoral School

October 20th-22nd, 2020

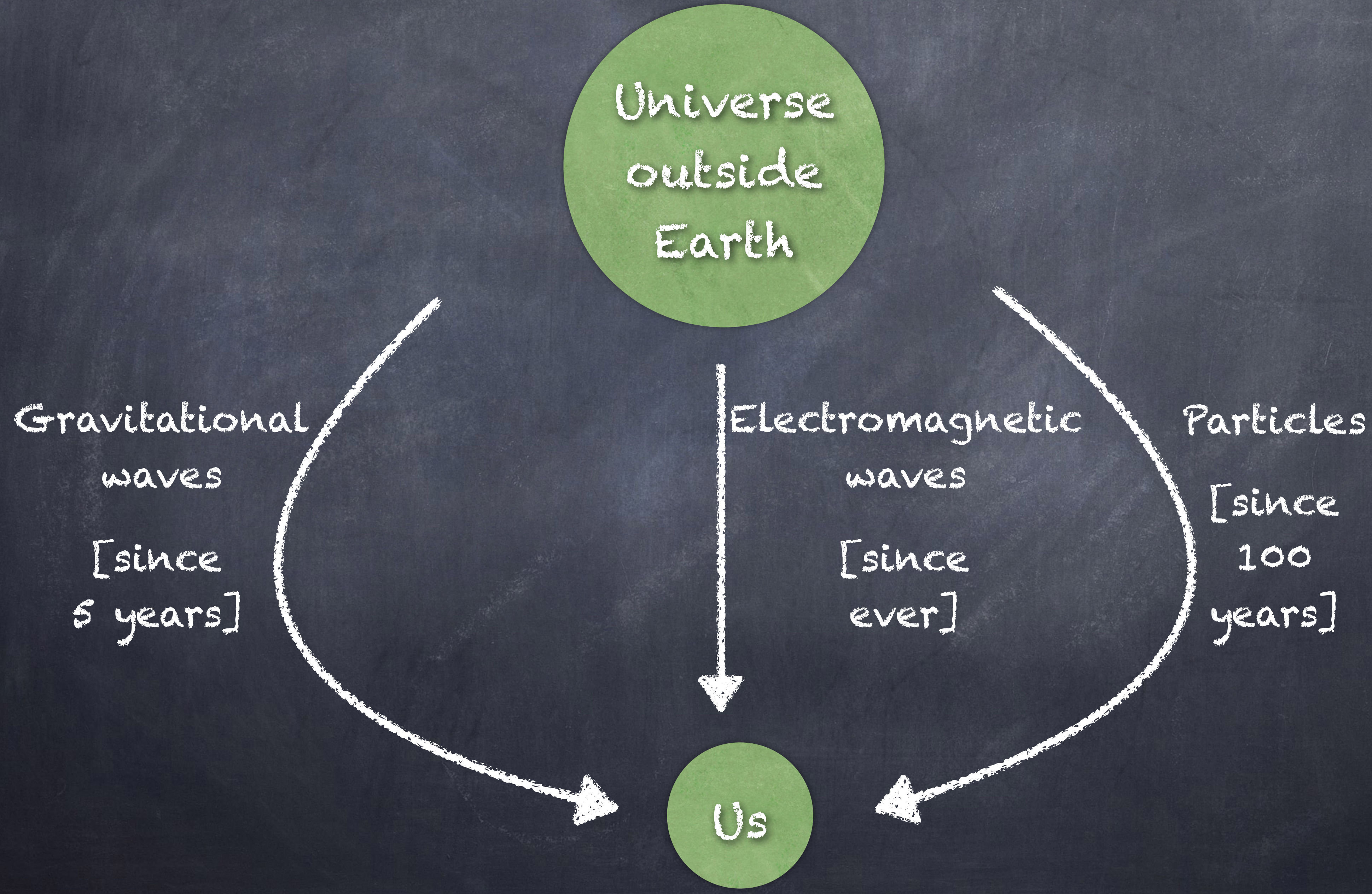
Exciting times...

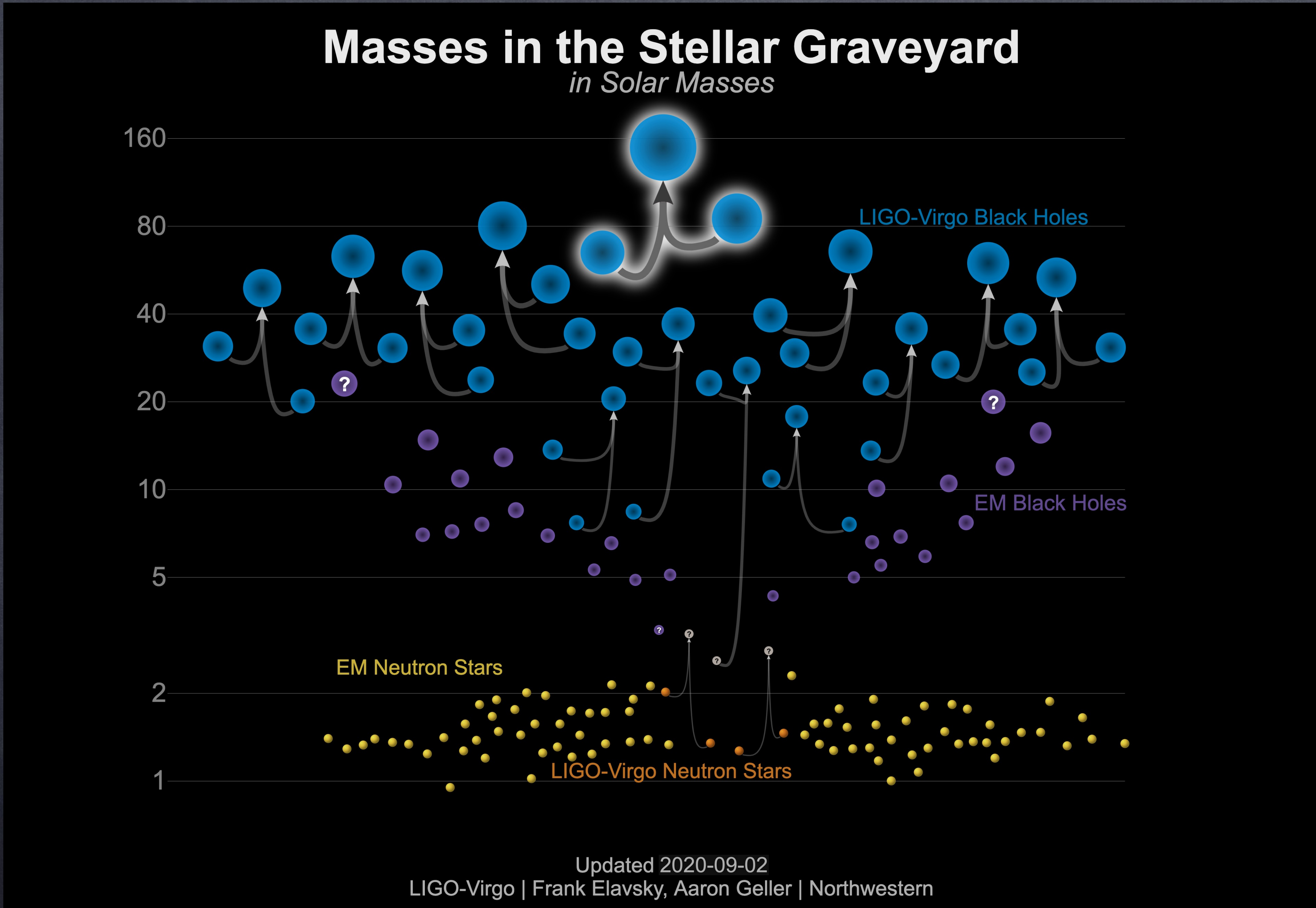
- 1915 Birth of General Relativity and in 1916: prediction of Gravitational Waves
- 1974 Hulse-Taylor pulsar :
  First indirect detection of GW



- 2015 First direct detection of binary black hole merger: event GW150914
- 2017 First direct multi-messenger detection of binary neutron star merger: event GW170817

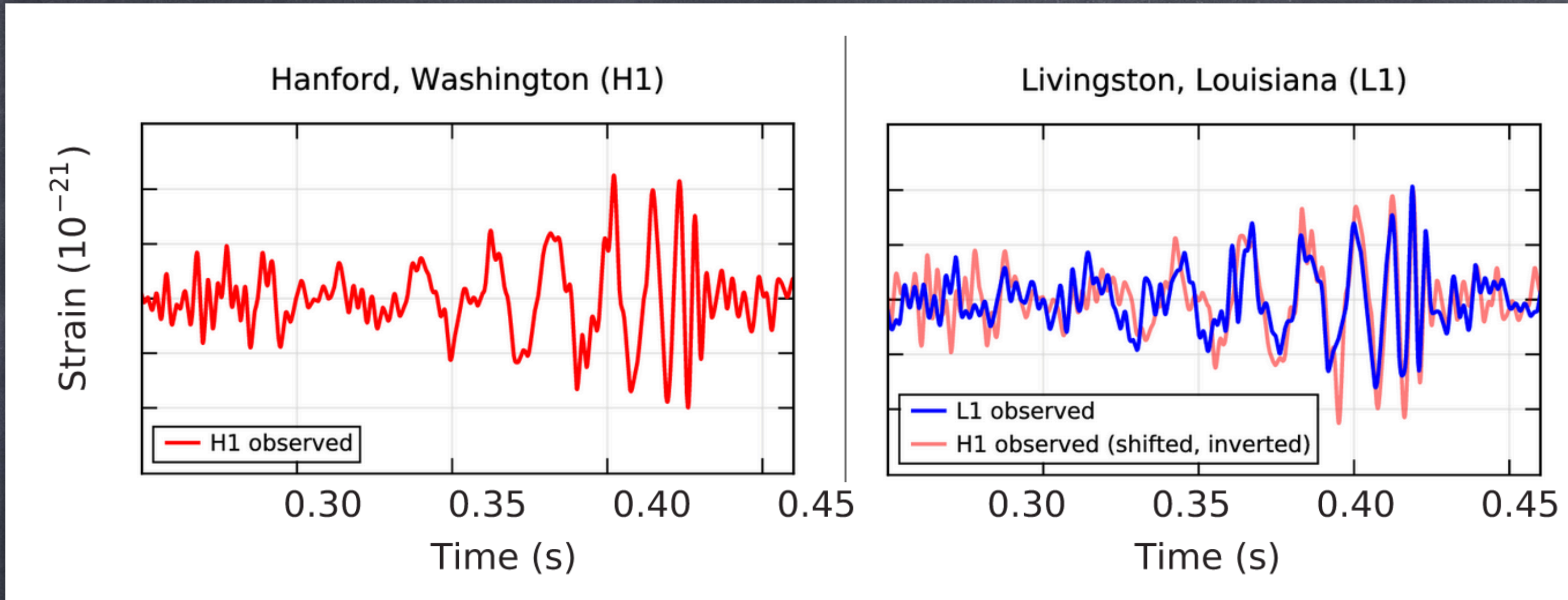
The three messengers





Masses in the Stellar Graveyard
in Solar Masses
160
80
40
20
10
5
2
1
LIGO-Virgo Black Holes
EM Black Holes
EM Neutron Stars
LIGO-Virgo Neutron Stars
Updated 2020-09-02
LIGO-Virgo | Frank Elavsky, Aaron Geller | Northwestern

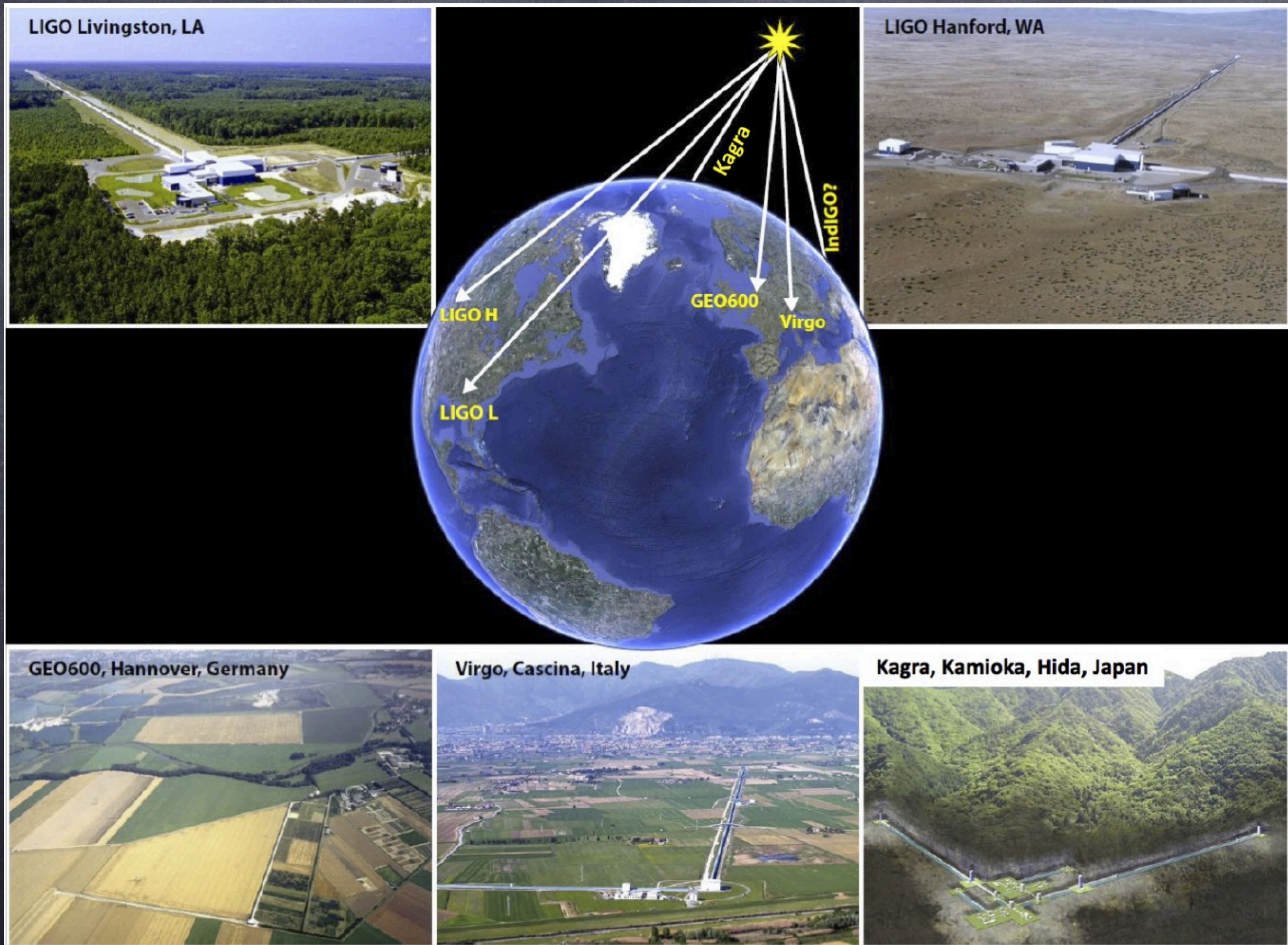
GW150914



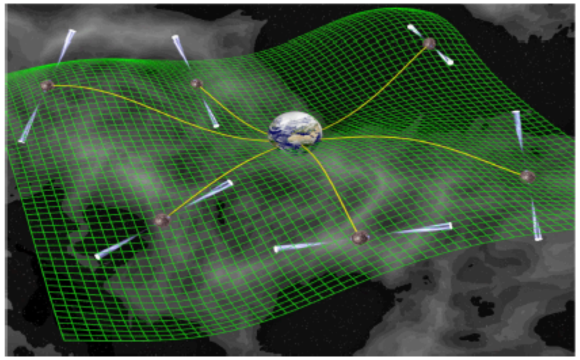
Estimated source parameters

Quantity	Value	Upper/Lower error estimate	Unit
Primary black hole mass	**36.2**	+5.2 -3.8	M sun
Secondary black hole mass	**29.1**	+3.7 -4.4	M sun
Final black hole mass	**62.3**	+3.7 -3.1	M sun
Final black hole spin	**0.68**	+0.05 -0.06	
Luminosity distance	**420**	+150 -180	Mpc
Source redshift, *z*	**0.09**	+0.03 -0.04	
Energy radiated	**3.0**	+0.5 -0.5	M sun

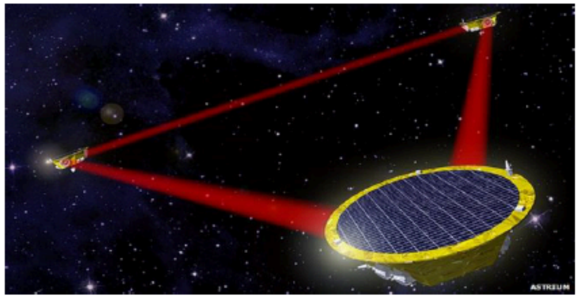
Current GW detectors



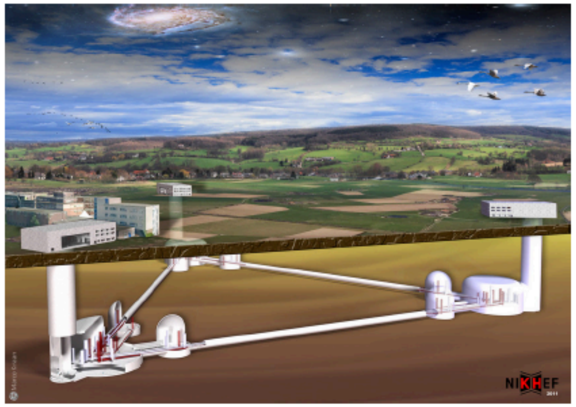
Upcoming detectors



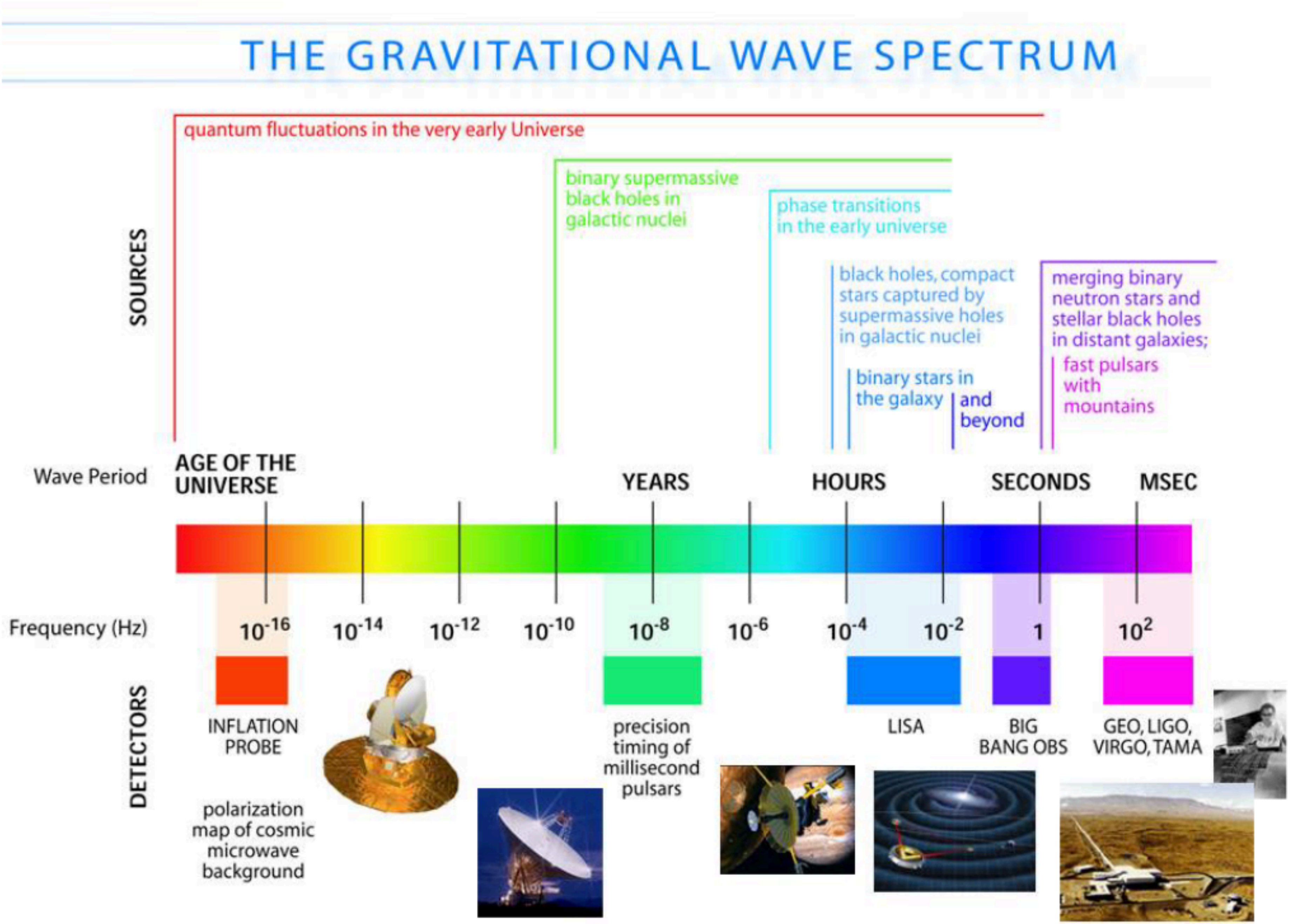
- Pulsar timing arrays



- Laser Interferometer Space Antenna (LISA)



- Einstein Telescope


THE GRAVITATIONAL WAVE SPECTRUM
SOURCES
quantum fluctuations in the very early Universe
binary supermassive black holes in galactic nuclei
phase transitions in the early universe
black holes, compact stars captured by supermassive holes in galactic nuclei
binary stars in the galaxy
and beyond
merging binary neutron stars and stellar black holes in distant galaxies;
fast pulsars with mountains
Wave Period
AGE OF THE UNIVERSE
YEARS
HOURS
SECONDS
MSEC
Frequency (Hz)
10^-16
10^-14
10^-12
10^-10
10^-8
10^-6
10^-4
10^-2
1
10^2
DETECTORS
INFLATION PROBE
polarization map of cosmic microwave background
precision timing of millisecond pulsars
LISA
BIG BANG OBS
GEO, LIGO, VIRGO, TAMA

GW and string theory

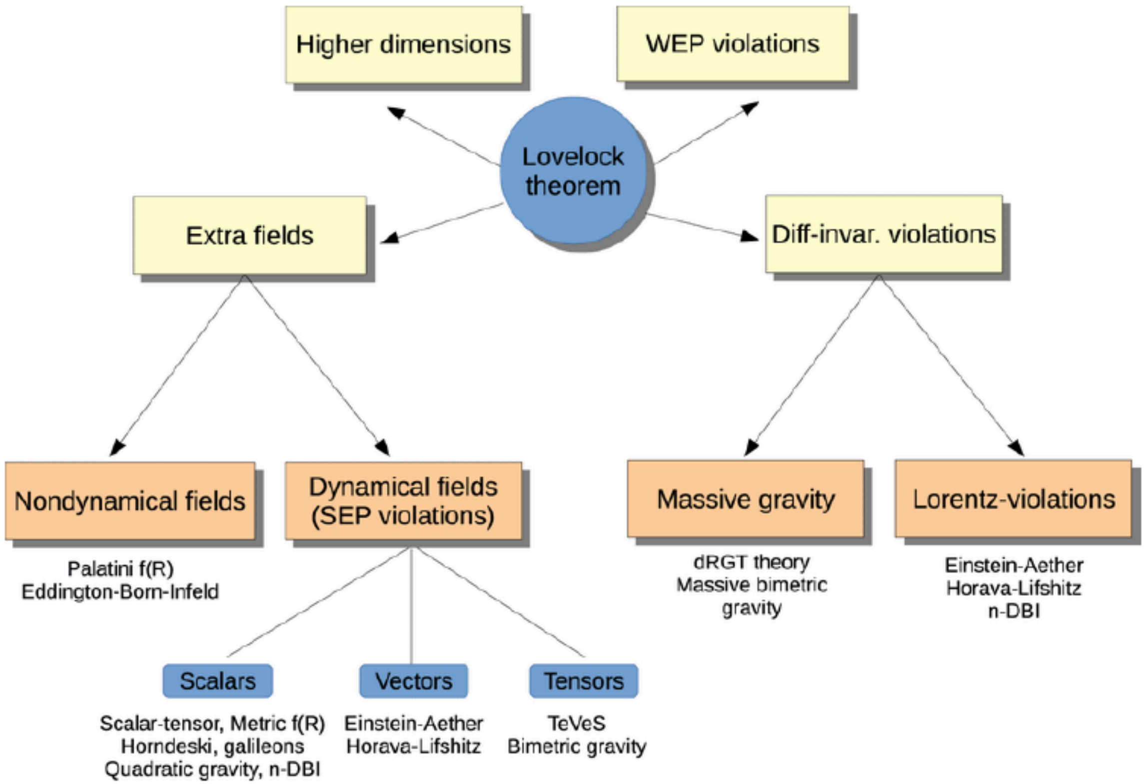
- Gap between Planck scale and observations
- Effective field theory approach: GR+suppressed modifications

Modifications of GR

- Lovelock's theorem:

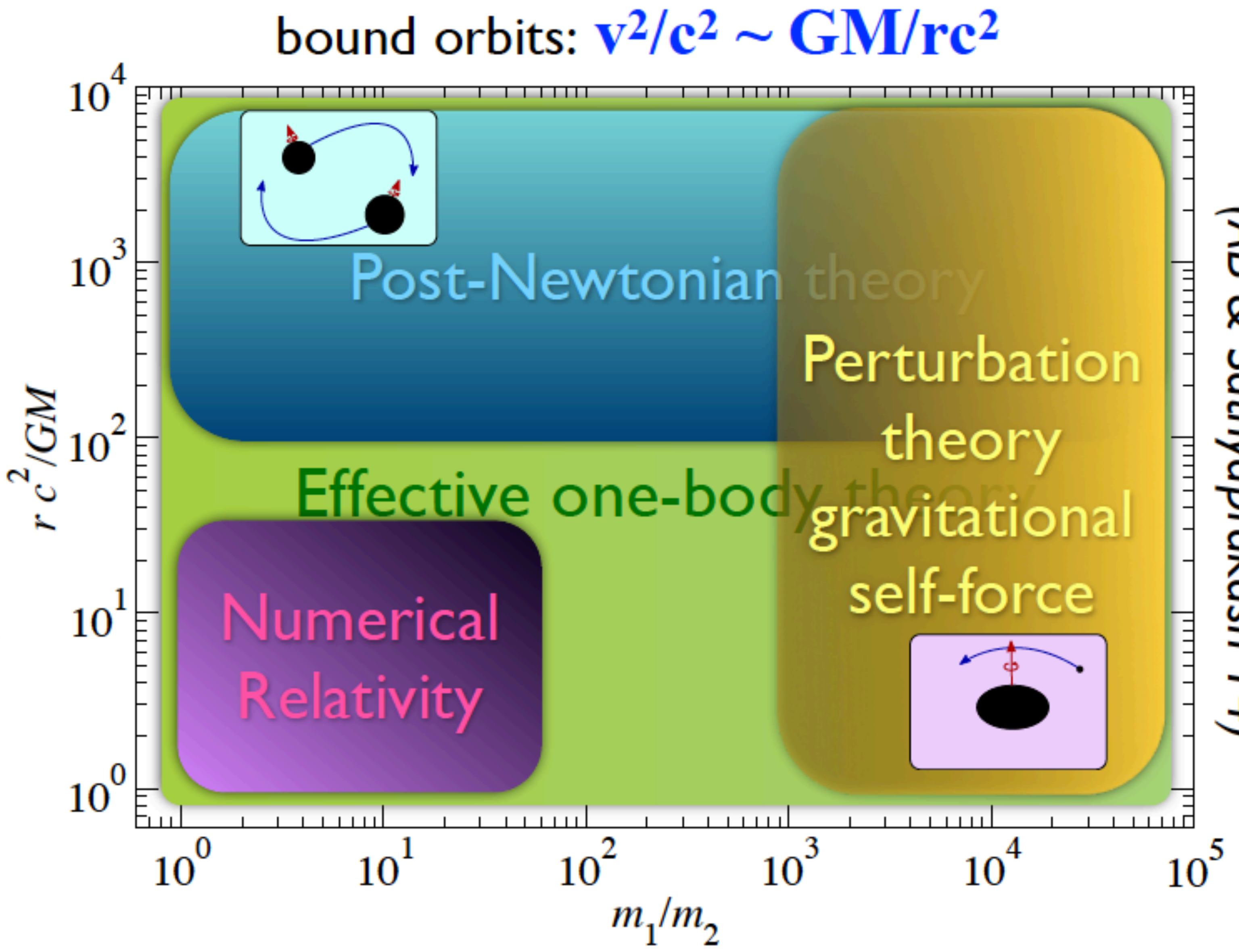
  "In four spacetime dimensions the only divergence-free symmetric rank-2 tensor constructed solely from the metric $g_{\mu\nu}$ and its derivatives up to second differential order, and preserving diffeomorphism invariance, is the Einstein tensor plus a cosmological term."

- Relaxing one or more of the assumptions allows for a plethora of alternative theories:



Berti et al., CQG **32**, 243001 (2015)

These lectures: two-body problem in GR



(AB & Sathyaprakash 14)

Outline of these lectures

1. Post-Newtonian/Post-Minkowksian theory
   1.1. Propagation, linear GR theory
   1.2. Interaction with test masses, freely falling frame, TT and detector frames, energy of GW
   1.3. Generation of GW
   1.4. Quasi-circular inspiral of compact binaries
2. Black hole Perturbation theory
   2.1. Mathisson-Papapetrou-Dixon theory
   2.2. Quasi-normal modes - Black Hole Spectroscopy

References

1. M. Maggiore, Gravitational Waves, Volume 1. Sections 1, 3 and 4.
2. L. Blanchet, Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries, 1310.1528
3. L. Barack and A. Pound, Self-force and radiation reaction in general relativity, 1805.10385
4. A. Harte, Motion in classical field theories and the foundations of the self-force problem, 1405.5077
5. E. Berti, V. Cardoso, A. Starinets, 0905.2975
6. G. Compère, A. Fiorucci, 1801.07064

1.1. Propagation, linear GR theory

General Relativity

$$S = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R[g] + S_M$$

Minimal coupling $\eta_{\mu\nu} \to g_{\mu\nu}$, $\partial_\mu \to D_\mu$

$$T^{\mu\nu} = \frac{2c}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}} \qquad G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Diffeomorphism invariance

$$x^\mu \to x'^\mu(x) \qquad g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x(x'))$$

Perturbation theory

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

Residual gauge transformations:

$$x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x) \qquad |\partial_\mu \xi_\nu| \ll 1$$

$$h_{\mu\nu}(x) \mapsto h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

Symmetries:

$$\xi_\mu = b_\mu$$
$$\xi_\mu = a_{[\mu\nu]} x^\nu$$

⟶ $h_{\mu\nu}$ invariant under Poincaré transformations

Linearized Riemann

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho})$$

Exercise: Prove it!

We define

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}$$

Trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

Note

$$\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - 2h = -h$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

Exercise

Prove

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho})$$

Remember

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma})$$

$$R^\alpha_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha_{\beta\delta} + \Gamma^\alpha_{\epsilon\gamma}\Gamma^\epsilon_{\beta\delta} - (\gamma \leftrightarrow \delta)$$

White board

Linearized Einstein

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

We can use the gauge freedom to go to harmonic / de Donder gauge / Lorenz gauge

$$\partial^\mu\bar{h}_{\mu\nu} = 0$$

Indeed, under $x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x)$

$$\bar{h}_{\mu\nu} \mapsto \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho)$$

$$\partial^\nu\bar{h}_{\mu\nu} \mapsto \partial^\nu\bar{h}_{\mu\nu} - \Box\xi_\mu$$

If originally $\partial^\nu\bar{h}_{\mu\nu} = f_\mu(x)$ we need to solve $\Box\xi_\mu = f_\mu(x)$ to reach harmonic gauge. This is always possible.

In harmonic gauge

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Acting with ∂^ν we get by consistency $\partial^\nu T_{\mu\nu} = 0$

- This equation is important both for generation of GW and for propagation

Outside sources,

$$\Box \bar{h}_{\mu\nu} = 0 = (-\frac{1}{c^2}\partial_t^2 + \nabla^2)\bar{h}_{\mu\nu}$$

→ GW propagate at light speed

TT gauge

For ξ^μ harmonic $\Box\xi^\mu = 0$ then $\xi_{\mu\nu} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho$ is harmonic.

⟶ Out of the 10 components of $\bar{h}_{\mu\nu}$ only 2 propagate as a wave.

We choose ξ^0 such that $\bar{h} = 0$

Proof: $tr(\bar{h}_{\mu\nu} + \xi_{\mu\nu}) = \bar{h} + 2\partial_\alpha \xi^\alpha - 4\partial_\alpha \xi^\alpha$

Solve the ODE $\partial_0 \xi^0 + \partial_i \xi^i = \frac{1}{2}\bar{h}$ for $\xi^0(x^0, x^i)$

Residual gauge parameter: $\xi^0(x^i)$ harmonic

It implies

$$\bar{h}_{\mu\nu} = h_{\mu\nu}$$

TT gauge

For ξ^μ harmonic $\Box\xi^\mu = 0$ then $\xi_{\mu\nu} \equiv \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho$ is harmonic.

$\longrightarrow$ Out of the 10 components of $\bar{h}_{\mu\nu}$ only 2 propagate as a wave.

We choose ξ^i such that $h^{0i} = 0$

Proof: $h_{0i} + \xi_{0i} = h_{0i} + \partial_0\xi_i + \partial_i\xi_0$

Solve the ODE $\partial_0\xi^i = -h_{0i} - \partial_i\xi_0$ for $\xi^i(x^0, x^j)$

Residual gauge parameter: $\xi^i(x^j)$ harmonic

It implies

$$h_{0i} = 0$$

Harmonic gauge reduces to

$$\mu = 0 \rightarrow \partial^0 h_{00} + \partial^i h_{0i} = \partial^0 h_{00} = 0 \longrightarrow h_{00}(\vec{x})$$

$$\mu = i \rightarrow \partial^j h_{ij} = 0$$

Newtonian potential
that is irrelevant for propagation
We set it to zero

$$h_{0\mu} = 0, \qquad h_{ii} = 0, \qquad \partial^i h_{ij} = 0$$

We reached TT gauge. Note that it cannot be reached inside of sources because $\Box \bar{h}^{\mu\nu} \neq 0$

We write a decomposition in plane waves in terms of the polarisation tensor

$$h_{ij}^{TT}(x) = e_{ij}(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$k^\mu = (\frac{\omega}{c}, \vec{k}), \qquad \frac{\omega}{c} = |\vec{k}|, \qquad \hat{n} = \frac{\vec{k}}{|k|}$$

Polarized GW

Choose $\hat{n}$ along the z axis

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

This gives the linear solution

$$ds^2 = -c^2 dt^2 + dz^2 + (1 + h_+ \cos[\omega(t - \frac{z}{c})])dx^2$$
$$+(1 - h_+ \cos[\omega(t - \frac{z}{c})])dy^2 + 2h_\times \cos[\omega(t - \frac{z}{c})]dxdy$$

TT Projector

Given a plane wave $h_{\mu\nu}(x)$ propagating along the direction $\hat{n}$ already in harmonic gauge but not yet in TT gauge, we reach TT gauge as

$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}$$

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

Exercise

Prove the following properties of

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

(i) projector $\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$

(ii) transverse $n^i\Lambda_{ij,kl} = 0 = n^k\Lambda_{ij,kl}$

(iii) traceless $\Lambda_{ii,kl} = 0 = \Lambda_{ij,kk}$

(iv) harmonic $\Box h_{ij} = 0 = \partial^k \bar{h}_{kl} \Rightarrow \quad \Box(\Lambda_{ij,kl}h_{kl}) = 0$

$$\partial^j(\Lambda_{ij,kl}h_{kl}) = 0$$

White board

Exercise

Prove the following properties of

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

(i) projector $\quad \Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$

(ii) transverse $\quad n^i \Lambda_{ij,kl} = 0 = n^k \Lambda_{ij,kl}$

(iii) traceless $\quad \Lambda_{ii,kl} = 0 = \Lambda_{ij,kk}$

(iv) harmonic $\quad \Box h_{ij} = 0 = \partial^k \bar{h}_{kl} \Rightarrow \quad \Box(\Lambda_{ij,kl}h_{kl}) = 0$

$$\partial^j(\Lambda_{ij,kl}h_{kl}) = 0$$

Hints:

$P_{ii} = 2$

$\Box n_i = -2n_i$

$\partial_a n_i \partial_b n_i = \gamma_{ab} \qquad \partial_i = n_i \partial_r + \partial_i \theta^a \partial_a$

$\partial_i \theta^a \partial_a n_j = P_{ij}$

$n_i P_{ij} = 0$

White board

1.2. Interaction with test masses, freely falling frame, TT and detector frames, energy of GW

Interaction of GW with test masses

- Detectors idealized as test masses
- Computations done in a reference frame. Physics invariant under the choice
- GW are simple in TT gauge. It corresponds to a specific reference frame/observer
- Detector is more intuitive in another frame, the detector proper frame
- We need to switch between the two frames
- Physical intuition comes from the geodesic deviation equation

Local inertial frame

- It is always possible to choose coordinates such that the Christoffel symbols vanish at one point
- The resulting coordinate system around that point is called a local inertial frame

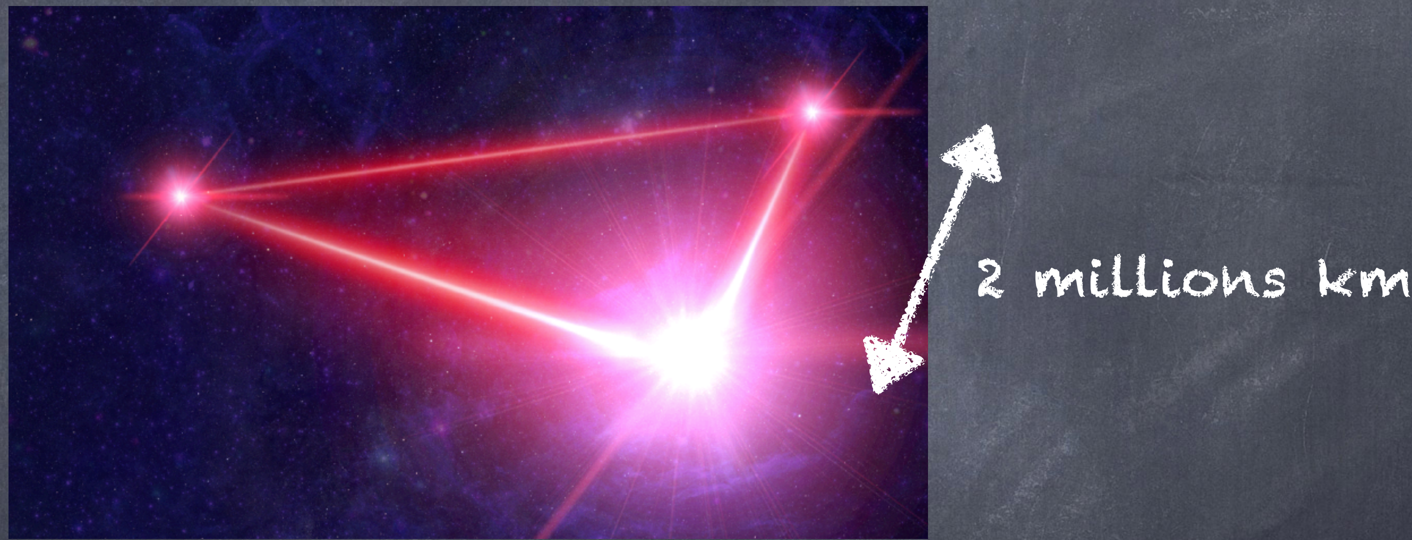
Freely falling frame

- It is always possible to choose coordinates such that on an entire timeline geodesic, all Christoffel symbols vanish
- For each point P in this frame, the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2}|_P = 0.$$

- In this frame, a test mass is freely falling. It gives a realization of the equivalence principle.

Laser Interferometer Space Antenna (LISA)



3 drag-free satellites:
Spacecrafts that adjust their position with thrusters in order to remain centred about a freely floating mass

To be launched in 2034

Fermi normal coordinates

A freely falling frame naturally define a special coordinate system around the test mass located at (t,0,0,0).
It is called a Fermi normal coordinate system.

Since

$$\Gamma^{\alpha}_{\beta\gamma}|_{\gamma} = 0$$

the metric around the test mass has no linear term in x^i

There are quadratic terms proportional to the Riemann tensor,

$$ds^2 \approx -c^2 dt^2 [1 + R_{0i0j} x^i x^j] - 2c dx^i dt (\frac{2}{3} R_{0jik} x^j x^k) + dx^i dx^j [\delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l]$$

Geodesic equation

Consider a curve $x^\mu(\lambda)$

The interval ds between 2 points separated by $d\lambda$ is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda^2$$

The velocity is $u^\mu = \dfrac{dx^\mu}{d\lambda}$

For a timelike curve the proper time τ is defined as

$$c^2 d\tau^2 = -ds^2 = -g_{\mu\nu} u^\mu u^\nu d\lambda^2$$

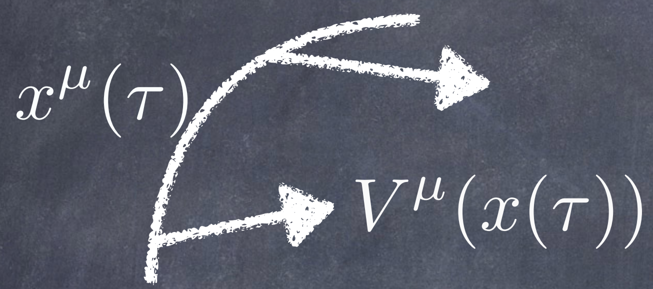
or after using $\lambda = \tau$ $\quad g_{\mu\nu} u^\mu u^\nu = -c^2$

The classical trajectory of a particle test mass m is obtained by extremizing the action

$$S = -m \int_{\tau_i}^{\tau_f} d\tau \quad \longrightarrow \quad \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x) u^\nu u^\rho = 0$$

Parallel transport

Consider a geodesics with proper time τ



We introduce the covariant derivative along the curve $x^\mu(\tau)$

$$\frac{DV^\mu}{D\tau} \equiv \frac{\partial V^\mu}{\partial \tau} + \Gamma^\mu_{\nu\rho} V^\nu \frac{dx^\rho}{d\tau}$$

Property: $\dfrac{DV^\mu}{D\tau}$ transforms as a vector.

The vector is parallelly transported along the curve.

Geodesic deviation

Now consider 2 geodesics, each one with proper time τ

$$\frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x+\xi)\frac{d(x^\nu + \xi^\nu)}{d\tau}\frac{d(x^\rho + \xi^\rho)}{d\tau} = 0$$

If $|\xi^\mu|$ is negligible with respect to the variation of the gravitational field, we can expand at first order in $\xi^\mu(\tau)$

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

Exercise

Prove: $$\frac{D^2\xi^\mu}{D\tau^2} = -\xi^\sigma R^\mu{}_{\nu\sigma\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

Geodesic deviation

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{D}{D\tau}\left(\frac{\partial\xi^\mu}{\partial\tau} + \Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau}\right)$$

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{\partial^2\xi^\mu}{\partial\tau^2} + \frac{dx^\lambda}{d\tau}\partial_\lambda\Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau} + \Gamma - terms$$

Use $$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$\frac{D^2\xi^\mu}{D\tau^2} = -\xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \frac{dx^\lambda}{d\tau}\partial_\lambda\Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau} + \Gamma - terms$$

$$\frac{D^2\xi^\mu}{D\tau^2} = \xi^\sigma\left(\partial_\nu\Gamma^\mu_{\sigma\rho} - \partial_\sigma\Gamma^\mu_{\nu\rho}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma - terms$$

$\longrightarrow$ $$\frac{D^2\xi^\mu}{D\tau^2} = -\xi^\sigma R^\mu{}_{\nu\sigma\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

TT frame

This is the coordinate frame in which the metric is in TT gauge

Consider a test mass initially at rest at $\tau = 0$.

The geodesic equation is

$$\frac{d^2x^i}{d\tau^2}|_{\tau=0} = -[\Gamma^i_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}]_{\tau=0}$$

$$= -[\Gamma^i_{00}(\frac{dx^0}{d\tau})^2]$$

because

$$\frac{dx^i}{d\tau}|_{\tau=0} = 0$$

At linear order, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ after taking Cartesian coordinates for the background.

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho})$$

$$\Gamma^{i}_{00} = \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00}) = 0$$

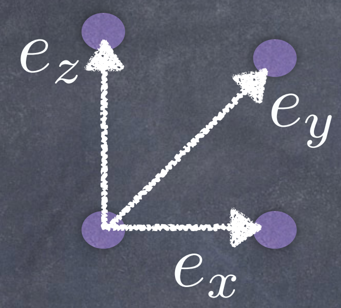
in TT gauge. Therefore,

$$\frac{d^2 x^i}{d\tau^2}\Big|_{\tau=0} = 0$$

In TT frame, particles which were at rest before the arrival of the wave remain at rest even after the arrival of the wave!
(non-linear effects are highly subleading)

The coordinates stretch themselves so that the position of the free test masses do not change

We can define the coordinates using freely falling test masses



What about time? In TT gauge, $h_{00} = h_{0i} = 0$

The proper time along a timelike trajectory $x^\mu(\tau)$ is obtained from

$$c^2 d\tau^2 = c^2 dt^2 - (\delta_{ij} + h_{ij}^{TT}) \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} d\tau^2$$

For a test mass initially at rest, $\frac{dx^i}{d\tau} = 0 \qquad \forall \tau$

→ $\tau = t$

In TT gauge, the proper time of a free test mass initially at rest is the coordinate time

Of course, proper distances change!

Consider the distance between $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$

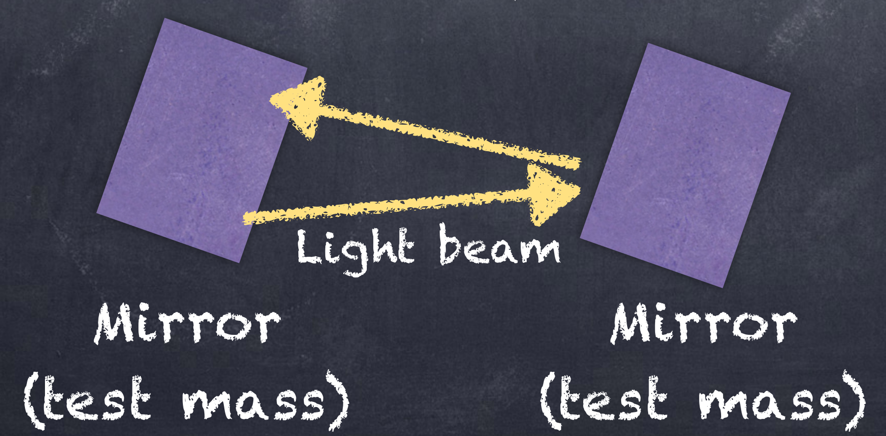
The coordinate distance $L = x_2 - x_1$ is a constant

But the proper distance after the passage of the wave is

$$s = \int_1^2 \sqrt{g_{xx}} dx = \sqrt{g_{xx}}(x_2 - x_1)$$

$$= \sqrt{1 + h_+ \cos(\omega t)} L \approx L(1 + \frac{1}{2} h_+ \cos(\omega t))$$

This is the basis of interferometry



Earth detector frame (LIGO/Virgo)

- Distance computed using rigid rulers
- No free fall with respect to the local inertial frame

$\vec{a} = -\vec{g}$ is the acceleration of the laboratory

- Local rotation with respect to local gyroscopes (e.g. Foucault pendulum)

$\vec{\Omega}$ is the angular velocity of the laboratory with respect to local gyroscopes.

Earth detector frame (LIGO/Virgo)

Result up to O(r^2) is $\qquad r = \sqrt{x^i x^i}$

inertial acc. — grav. redshift — Lorentz time dil.

$$ds^2 \approx -c^2 dt^2 [1 + \frac{2}{c^2}\vec{a}\cdot\vec{x} + \frac{1}{c^4}(\vec{a}\cdot\vec{x})^2 - \frac{1}{c^2}(\vec{\Omega}\times\vec{x})^2 + R_{0i0j}x^i x^j]$$

GW and varying grav.

$$+ 2c dt dx^i [\frac{1}{c}\epsilon_{ijk}\Omega^j x^k - \frac{2}{3}R_{0ijk}x^j x^k]$$

Sagnac effect — GW and varying grav.

$$+ dx^i dx^j [\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l]$$

GW and varying grav.

This is the detector frame used by experimentalists on Earth

We denote by L_B the typical variation scale of the metric

$$\frac{\vec{a}}{c^2} \sim \frac{1}{L_B}, \qquad R_{\mu\nu\rho\sigma} \sim \frac{1}{L_B^2}$$

Earth detector frame (LIGO/Virgo)

$$ds^2 \approx -c^2dt^2[1 + \frac{2}{c^2}\vec{a}\cdot\vec{x} + \frac{1}{c^4}(\vec{a}\cdot\vec{x})^2 - \frac{1}{c^2}(\vec{\Omega}\times\vec{x})^2 + R_{0i0j}x^ix^j]$$
$$+ 2cdtdx^i[\frac{1}{c}\epsilon_{ijk}\Omega^jx^k - \frac{2}{3}R_{0ijk}x^jx^k]$$
$$+ dx^idx^j[\delta_{ij} - \frac{1}{3}R_{ikjl}x^kx^l]$$

We denote by L_B the typical variation scale of the metric

$$\frac{\vec{a}}{c^2} \sim \frac{1}{L_B}, \qquad R_{\mu\nu\rho\sigma} \sim \frac{1}{L_B^2}$$

At order $O(\frac{r}{L_B})^0$ ⟶ Minkowski (in contrast to TT gauge where there is no such expansion!)

$O(\frac{r}{L_B})^1$ ⟶ Newtonian corrections

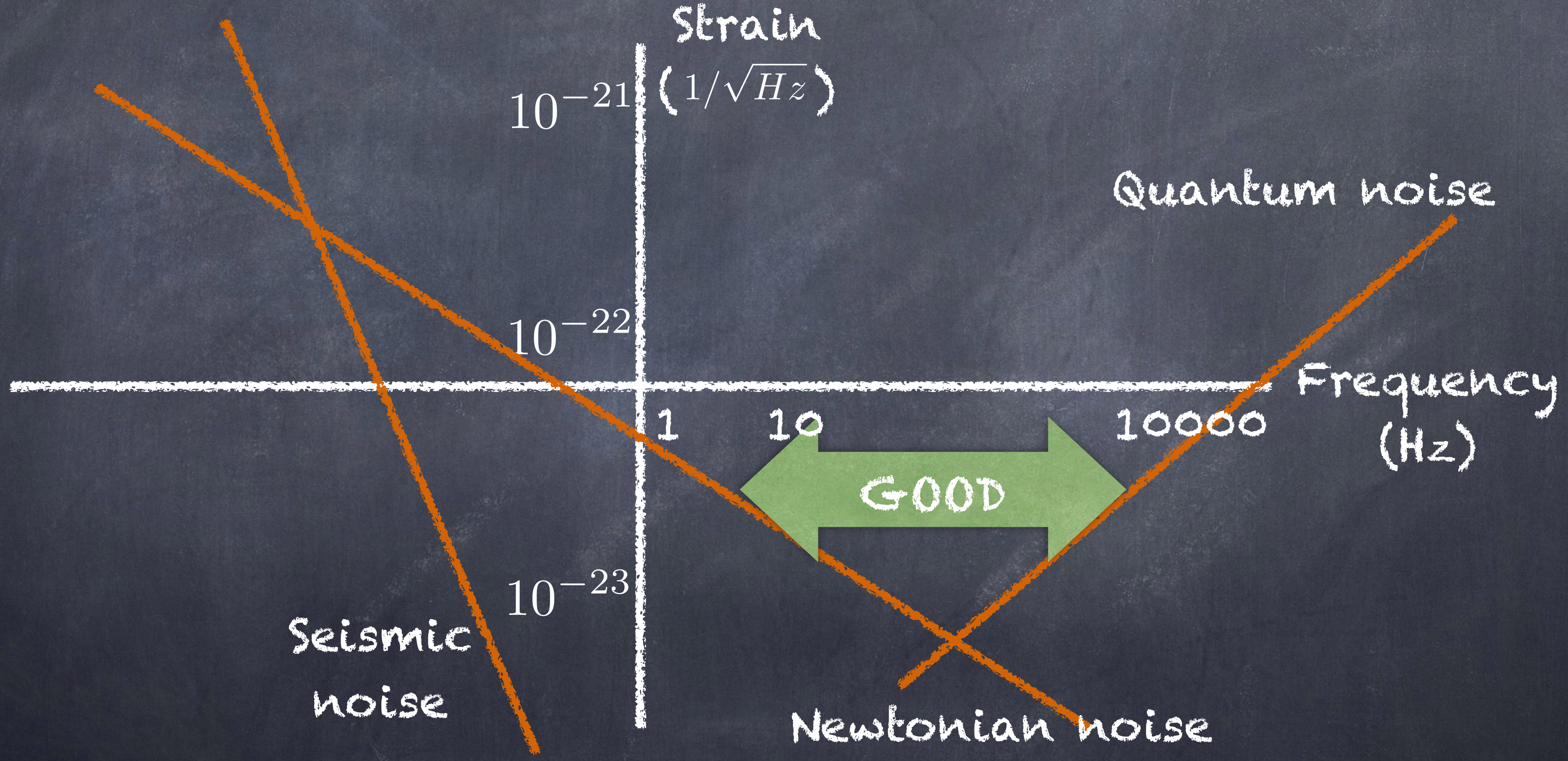
Geodesic equation: $\frac{d^2x^i}{d\tau^2} = -a^i - 2(\Omega\times v)^i + O(x^i)$

Earth gravity — Coriolis force

$O(\frac{r}{L_B})^2$ ⟶ GW and noise from varying gravity

In conclusion, a description in terms of Newtonian concept of forces is possible in the Earth detector frame.

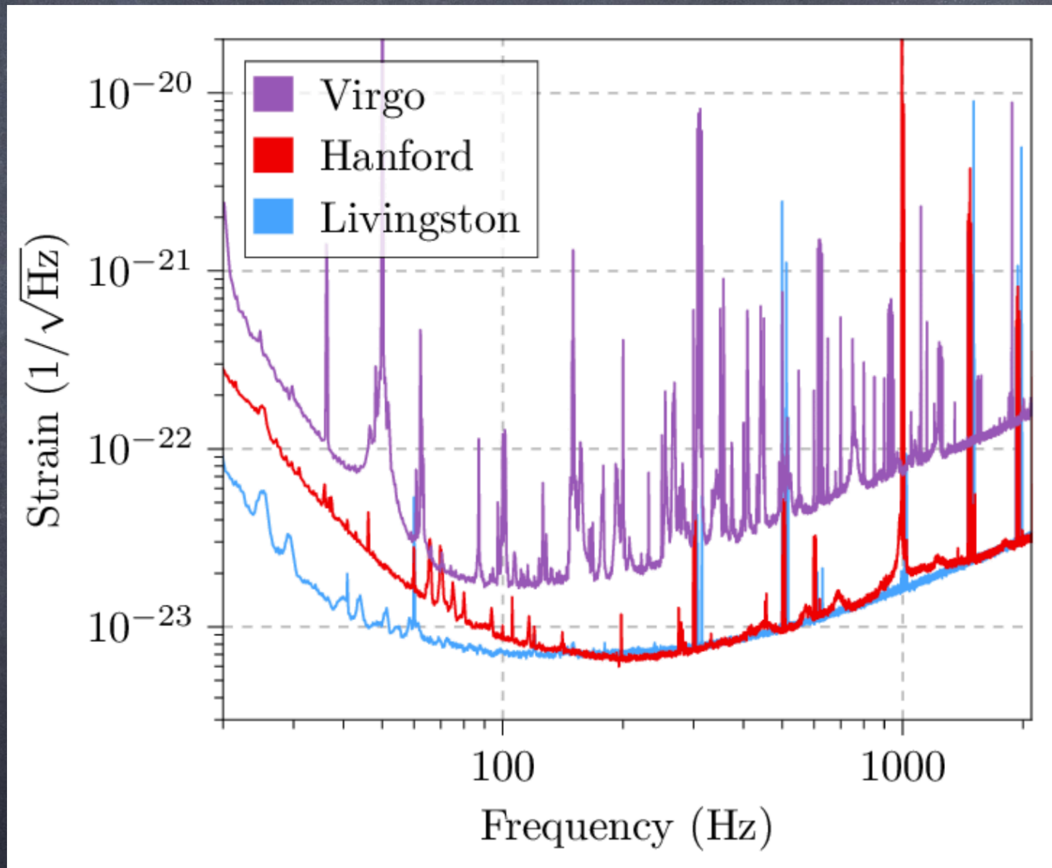
Graviational Waves are subleading with respect to uncertainties in Earth gravity and local rotation, Newtonian and seismic noise.



Einstein Telescope: 2 possible sites: Sardaigna (better seismic noise) and Euregio (better Newtonian noise)

In conclusion, a description in terms of Newtonian concept of forces is possible in the Earth detector frame.

Gravitational Waves are subleading with respect to uncertainties in Earth gravity and local rotation, Newtonian and seismic noise.



To isolate GW, we focus on detector in its frequency window. Acceleration is compensated by suspenders.

Only Riemann terms matter and the expression reduces to $x^{\mu}(\tau)$ freely falling frame, as far as we only look at components of in the direction unconstrained by the suspension mechanism

If we look at the equation for geodesic deviation at the centre P of the local inertial frame

$$\Gamma^{\alpha}_{\mu\nu}|_P = 0 \qquad \frac{D^2\xi^i}{D\tau^2} = \frac{d^2\xi^i}{d\tau^2} = -R^i{}_{0j0}\xi^j\left(\frac{dx^0}{d\tau}\right)^2$$

Non relativistic motion $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \approx c$

$$\longrightarrow \quad \ddot{\xi}^i = -c^2 R^i{}_{0j0}\xi^j$$

$$= -c^2\left(-\frac{1}{2c^2}\ddot{h}^{TT}_{ij}\right)\xi^j$$

We can compute the Riemann in any frame including TT gauge.

Finally, in Earth detector frame in the directions unconstrained by suspenders:

$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j$$

In the Earth detector frame, the effect of GW on a point particle of mass m is described as a Newtonian force

$$F_i = \frac{m}{2}\ddot{h}_{ij}^{TT}\xi^j$$

without reference to GR.

Note that here we assumed $|\xi^i| \ll L_B$ which is true for LIGO.

Motion of test masses

- We consider a ring of test masses initially at rest in the Earth detector frame.
- We fix the origin at the centre of the ring
- Then ξ^i describes the distance w.r.t the origin (coordinate distance = proper distance)
- We consider GW propagating in the z direction. The ring is in the (x,y) plane

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j$$

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

↓ sin

$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j$$

- If the particle is at z=0 at t=0 it will remain there.
- Therefore, GW are transverse in their physical effect: they displace the test masses transversally w.r.t. their direction of propagation

We write

$$\xi_i(t) = (x_0 + \delta x(t), y_0 + \delta y(t), 0)$$

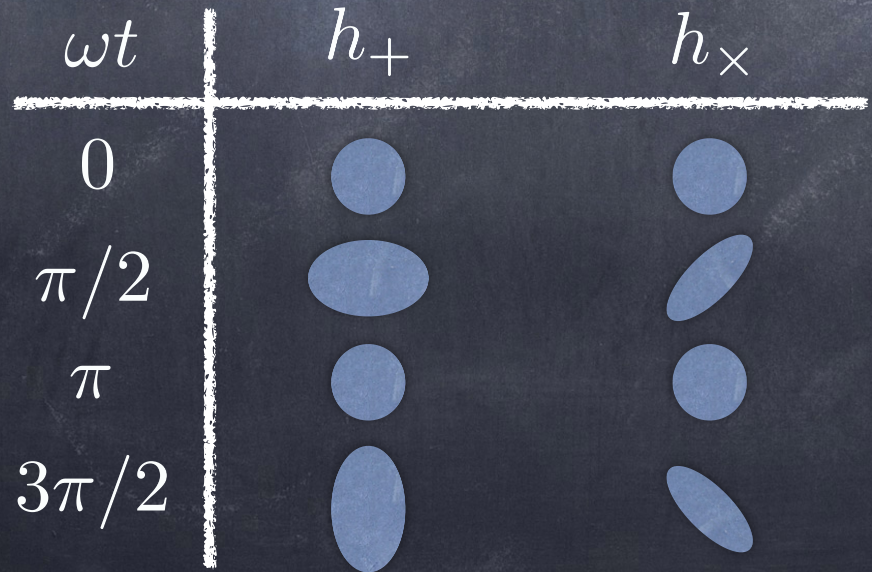
(x_0, y_0) initial position

$$\delta x \ll x_0, \qquad \delta y \ll y_0$$

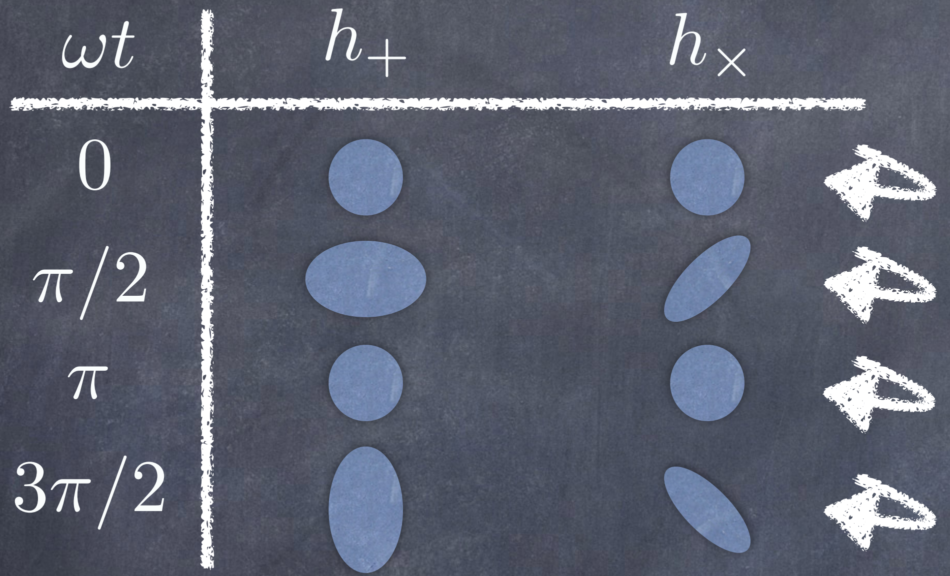
For h_+

$$\ddot{\delta x} = \frac{h_+}{2}(-\omega^2)\sin\omega t x_0 \qquad \longrightarrow \qquad \delta x = \frac{h_+}{2} x_0 \sin\omega\, t$$

$$\ddot{\delta y} = \frac{h_+}{2}\omega^2 \sin\omega t y_0 \qquad \qquad \delta y = -\frac{h_+}{2} y_0 \sin\omega\, t$$



Helicity of the graviton



Invariance under rotations of angle $\frac{2\pi}{2}$

Graviton has helicity 2

Proof:

$$h_{\mu\nu}(x) \mapsto h'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x) \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t-\frac{z}{c})] \quad \mapsto \quad h_{ij}^{TT}(t,z) = \begin{pmatrix} h'_+ & h'_\times & 0 \\ h'_\times & h'_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t-\frac{z}{c})]$$

$$h'_+ = h_+ \cos 2\psi - h_\times \sin 2\psi$$
$$h'_\times = h_+ \sin 2\psi + h_\times \cos 2\psi$$

$$h_\times \pm i h_+ \mapsto e^{\mp 2i\psi}(h_\times \pm i h_+)$$

helicity eigenstates

Energy of GW

- It is clear that GW carry energy-momentum: they can accelerate masses ! (Derived by Bondi in 1961)
- According to GR, any form of energy induces curvature
- GW then backreacts at second order beyond linear order and that curvature allows to define energy.

Consistency of perturbation theory

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots \qquad |h_{\mu\nu}| \ll 1$$

Consistent if backreaction is smaller than the perturbation (far from masses)

In general, close to masses that generate GW:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots \qquad |h_{\mu\nu}| \ll 1$$

We will distinguish the notion of background and perturbation by their frequency content :

background = low frequency
perturbation = high frequency

We consider the situation in which in some reference frame we can separate the metric into a background plus fluctuations where separation is based on a scale in time or space ("short-wave expansion")

$$\bar{\lambda} \ll L_B \qquad\qquad f \gg f_B$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Two small parameters: (1) $h \equiv O(h_{\mu\nu})$

(2) $\dfrac{\bar{\lambda}}{L_B}, \qquad \dfrac{f_B}{f}$

We now expand Einstein's equations to quadratic order in $h_{\mu\nu}$

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

$$R_{\mu\nu}[g] = \bar{R}_{\mu\nu}[\bar{g}] + R^{(1)}_{\mu\nu}[h;\bar{g}] + R^{(2)}_{\mu\nu}[h,h;\bar{g}] + \dots$$

low freq modes	high freq modes	both modes $e^{\vec{k}\cdot\vec{x}}e^{-\vec{k}\cdot\vec{x}} \sim e^0$

Low mode eqs: $\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{Low} + \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{low}$

High mode eqs: $R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{High} + \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{high}$

The low equation gives energy-momentum of GW. It can be obtained practically as follows. We introduce an intermediate time scale $\bar{t}$

$$\frac{1}{f_B} \gg \bar{t} \gg \frac{1}{f}$$

We then average over $\bar{t}$, i.e. over many periods of GW:

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{8\pi G}{c^4} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle$$

This was understood in the sixties. This is a renormalisation group flow. We integrated out high frequencies to describe physics of low frequencies.

We define the effective stress-tensor of GW

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G} \langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle \qquad R^{(2)} \equiv \bar{g}^{\mu\nu} R^{(2)}_{\mu\nu}$$

$$t = \bar{g}^{\mu\nu} t_{\mu\nu} = +\frac{c^4}{8\pi G} \langle R^{(2)} \rangle$$

because $\bar{g}_{\mu\nu}$ is low frequency

We deduce

$$-\langle R^{(2)}_{\mu\nu} \rangle = \frac{8\pi G}{c^4} (t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t)$$

We define

$$\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle \equiv \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T}$$

The effective Einstein's equations at low frequencies

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{8\pi G}{c^4} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle$$

become

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \frac{8\pi G}{c^4} (\bar{T}_{\mu\nu} + t_{\mu\nu})$$

This is the coarse-grained form of Einstein's equations

We equated different orders in h. This is possible because there is a second small parameter $\frac{\bar{\lambda}}{L_B} \ll 1$.

Explicit expressions:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\bar{D}^{\alpha}\bar{D}_{\mu}h_{\nu\alpha} + \bar{D}^{\alpha}\bar{D}_{\nu}h_{\mu\alpha} - \bar{D}^{\alpha}\bar{D}_{\alpha}h_{\mu\nu} - \bar{D}_{\nu}\bar{D}_{\mu}h\right)$$

$$\begin{aligned}R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[&\frac{1}{2}\bar{D}_{\mu}h_{\rho\alpha}\bar{D}_{\nu}h_{\sigma\beta} + (\bar{D}_{\rho}h_{\nu\alpha})(\bar{D}_{\sigma}h_{\mu\beta} - \bar{D}_{\beta}h_{\mu\sigma})\\ &+h_{\rho\alpha}(\bar{D}_{\nu}\bar{D}_{\mu}h_{\sigma\beta} + \bar{D}_{\beta}\bar{D}_{\sigma}h_{\mu\nu} - \bar{D}_{\beta}\bar{D}_{\nu}h_{\mu\sigma} - \bar{D}_{\beta}\bar{D}_{\mu}h_{\nu\sigma})\\ &+(\frac{1}{2}\bar{D}_{\alpha}h_{\rho\sigma} - \bar{D}_{\rho}h_{\alpha\sigma})(\bar{D}_{\nu}h_{\mu\beta} + \bar{D}_{\mu}h_{\nu\beta} - \bar{D}_{\beta}h_{\mu\nu})\Big]\end{aligned}$$

Einstein equations' constraints on the small parameters

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}(\bar{T}_{\mu\nu} + t_{\mu\nu})$$

We choose coordinates such that $\bar{g}_{\mu\nu} = O(1)$

$$\partial\bar{g}_{\mu\nu} \sim \frac{1}{L_B} \longrightarrow \bar{R}_{\mu\nu} \sim \partial^2\bar{g}_{\mu\nu} \sim \frac{1}{L_B^2}$$

$$\partial h_{\mu\nu} \sim \frac{h}{\bar{\lambda}} \longrightarrow R^{(2)}_{\mu\nu} \sim (\partial h)^2 \sim \frac{h^2}{\bar{\lambda}^2}$$

Einstein's equations give

$$\frac{1}{L_B^2} \sim \frac{h^2}{\bar{\lambda}^2} + (matter) \longleftrightarrow h \sim \frac{\bar{\lambda}}{L_B} \quad \text{if no matter}$$

If dominant matter $\frac{1}{L_B^2} \gg \frac{h^2}{\bar{\lambda}^2} \longleftrightarrow h \ll \frac{\bar{\lambda}}{L_B}$

Therefore, if background slow frequency perturbation are neglected, the perturbation series in h breaks down.

Gravitational energy -momentum tensor

The energy-momentum tensor can be computed assuming

$\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ (far from sources, ignoring Earth gravity)

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle$$

We do the computation in harmonic and $h = 0$ gauge

$$\partial_\mu h^{\mu\nu} = 0, \qquad h = 0$$

In harmonic gauge, a perturbation is a superposition of plane waves

$$h_{\mu\nu} = \int \epsilon_{\mu\nu} e^{i\omega(t - \frac{\hat{n}\cdot\vec{x}}{c})}, \qquad \vec{k} = \frac{\omega}{c}\hat{n}$$

We have

$$\partial_t h_{\mu\nu} = i\omega h_{\mu\nu}$$

$$\partial_i h_{\mu\nu} = -\frac{i\omega}{c} n_i h_{\mu\nu} \longrightarrow \partial_i h_{\mu\nu} = -\frac{1}{c} n_i \partial_t h_{\mu\nu}$$

Integrations by parts in $R^{(2)}_{\alpha\beta}$ are possible

∂_t total derivatives drop because of time average

∂_i can be expressed as ∂_t because waves

$$\longrightarrow \langle \partial_\alpha T^\alpha_{\mu\nu} \rangle = 0, \qquad \forall T^\alpha_{\mu\nu}$$

Exercise

Given

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

Assume $g_{\mu\nu} = \eta_{\mu\nu} \quad \partial_\mu h^{\mu\nu} = 0, \quad h = 0 \quad \Box h_{\alpha\beta} = 0$

Use $\langle \partial_\alpha T^\alpha_{\mu\nu} \rangle = 0, \quad \forall T^\alpha_{\mu\nu}$ **to prove**

(i) $\langle R_{(2)} \rangle = 0$

(ii) $\langle R^{(2)}_{\mu\nu} \rangle = -\frac{1}{4}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} \rangle$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle = 0$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle = 0$$

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2}\langle\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta}\partial^\alpha h^\beta_\nu - \partial_\alpha h_{\mu\beta}\partial^\beta h^\alpha_\nu$$
$$+h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - h^{\alpha\beta}\partial_\alpha\partial_\mu h_{\nu\beta} - h^{\alpha\beta}\partial_\alpha\partial_\nu h_{\mu\beta}\rangle$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle = 0$$

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2}\langle\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta}\partial^\alpha h^\beta_\nu - \partial_\alpha h_{\mu\beta}\partial^\beta h^\alpha_\nu$$
$$+h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - h^{\alpha\beta}\partial_\alpha\partial_\mu h_{\nu\beta} - h^{\alpha\beta}\partial_\alpha\partial_\nu h_{\mu\beta}\rangle$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle = 0$$

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2}\langle\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta}\partial^\alpha h^\beta_\nu - \partial_\alpha h_{\mu\beta}\partial^\beta h^\alpha_\nu$$
$$+h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - h^{\alpha\beta}\partial_\alpha\partial_\mu h_{\nu\beta} - h^{\alpha\beta}\partial_\alpha\partial_\nu h_{\mu\beta}\rangle$$

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$$

Gravitational energy -momentum tensor

Energy-momentum tensor is

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \longrightarrow t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$$

Do residual gauge transformation change the stress-tensor?

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$\delta_\xi t_{\mu\nu} = ?$$

Exercise

Prove $\delta_\xi t_{\mu\nu} = 0$

Using $t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Solution

Prove $\delta_\xi t_{\mu\nu} = 0$

Using $$t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Remember $$h = 0 \rightarrow \partial_\alpha \xi^\alpha = 0$$
$$\partial_\alpha h^{\alpha\mu} = 0 \rightarrow \Box \xi^\mu + \partial_\mu \partial_\alpha \xi^\alpha = 0$$

Therefore $$\langle \partial_\alpha (\partial_{...} h_{\mu\nu} \partial_{...} \xi^\sigma)\rangle = 0$$

Solution

Prove $\delta_\xi t_{\mu\nu} = 0$

Using $t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Remember

$$h = 0 \to \partial_\alpha \xi^\alpha = 0$$
$$\partial_\alpha h^{\alpha\mu} = 0 \to \Box \xi^\mu + \partial_\mu \partial_\alpha \xi^\alpha = 0$$

Therefore $\langle \partial_\alpha(\partial_{...} h_{\mu\nu} \partial_{...} \xi^\sigma)\rangle = 0$

Compute

$$\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu(\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha)\partial_\nu h^{\alpha\beta}\rangle + (\mu \leftrightarrow \nu)$$
$$\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu \partial_\alpha \xi_\beta \partial_\nu h^{\alpha\beta}\rangle + (\mu \leftrightarrow \nu)$$
$$\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu \xi_\beta \partial_\alpha \partial_\nu h^{\alpha\beta}\rangle + (\mu \leftrightarrow \nu) \sim 0$$

Gravitational energy -momentum tensor

Energy-momentum tensor is

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \longrightarrow t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$$

Question: do residual gauge transformation change the stress-tensor? Answer: no.

Therefore, we can replace $h_{\mu\nu}$ by $h^{TT}_{\mu\nu}$

In particular, the effective energy density is

$$t^{00} = \frac{c^2}{32\pi G}\langle \dot{h}^{TT}_{ij}\dot{h}^{TT}_{ij}\rangle = \frac{c^2}{16\pi G}\langle \dot{h}^2_+ + \dot{h}^2_\times\rangle$$

$$\dot{f} = \partial_t f = c\partial_0 f$$

$$h^{TT}_{ij} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

Bianchi identities imply

$$\bar{D}^\mu(\bar{T}_{\mu\nu} + t_{\mu\nu}) = 0$$

This proves that one can associate to each Poincaré symmetry a conserved quantity (up to higher order perturbations)

Far from the sources

$$\partial^\mu t_{\mu\nu} = 0$$

$$\longrightarrow \quad \int_V d^3x[\partial_0 t^{00} + \partial_i t^{i0}] = 0 \quad \text{on a given volume V}$$

The effective energy of the volume is

$$E_V \equiv \int d^3x t^{00}$$

$$\frac{1}{c}\frac{dE_V}{dt} = -\int_V d^3x \partial_i t^{0i} = -\int dA n_i t^{0i}$$

Take S a spherical surface $dA = r^2 d\Omega$

$$\frac{dE_V}{dt} = -c\,r^2 \int d\Omega\, t^{0r}$$

$$= \frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{TT} \frac{\partial}{\partial r} h_{ij}^{TT} \rangle$$

Single plane waves are not realistic as global solutions at large distances. Instead, a GW propagating radially outwards has the following form at large distances from the source (proven later)

$$h_{ij}^{TT} = \frac{1}{r} f_{ij}(t - \frac{r}{c}) + O(r^{-2})$$

This is similar to electromagnetic waves, up to a spin-2 polarization

Exercise

Prove that $h_{ij}^{TT} = \frac{1}{r} f_{ij}(t - \frac{r}{c})$ obeys $\Box h_{ij}^{TT} = O(r^{-2})$

Solution

Prove that $h_{ij}^{TT} = \frac{1}{r} f_{ij}(t - \frac{r}{c})$ obeys $\Box h_{ij}^{TT} = O(r^{-2})$

Use $\Box\Phi = \frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\mu \Phi \right)$

$$ds^2 = -c^2 dt^2 + dr^2 \qquad \sqrt{|g|} = c$$

$$ds^2 = -c^2 du^2 - 2c\,du\,dr \qquad \sqrt{|g|} = c$$

Drop c

$$x^\mu = (u, r) \qquad g^{\mu\nu} = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

$$V^\mu = g^{\mu\nu}\partial_\nu\left(\frac{f_{ij}(u)}{r}\right) = \left(-\frac{-1}{r^2} f_{ij}, -\left(\frac{1}{r}\right)\partial_u f_{ij} + \left(-\frac{1}{r^2}\right) f_{ij}\right)$$

$$\partial_\mu V^\mu = \frac{1}{r^2}\partial_u f_{ij} + \frac{1}{r^2}\partial_u f_{ij} + \frac{1}{r^3} f_{ij} = O(r^{-2})$$

$$\frac{\partial}{\partial r} h_{ij}^{TT} = -\frac{1}{r^2} f_{ij}(t - \frac{r}{c}) + \frac{1}{r}(-\frac{1}{c})\frac{\partial}{\partial t} f_{ij} = -\partial_0 h_{ij}^{TT} = \partial^0 h_{ij}^{TT}$$

$$\longrightarrow \quad t^{0r} = t^{00}$$

An observer sitting at large distances sees a plane wave front.

Since E_V decreases, GW carry away an energy flux

$$\frac{dE}{dAdt} = +c\,t^{00} = \frac{c^3}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle$$
$$= \frac{c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle$$

This is Einstein's formula for the flux-balance law of energy, with a factor of 2 corrected by Eddington.

Poincaré flux-balance laws

For each Killing vector of Minkowski, we have a conserved current :

$$J^\mu = t^{\mu\nu}\bar{\xi}_\nu \qquad\qquad \partial_\mu J^\mu = 0$$

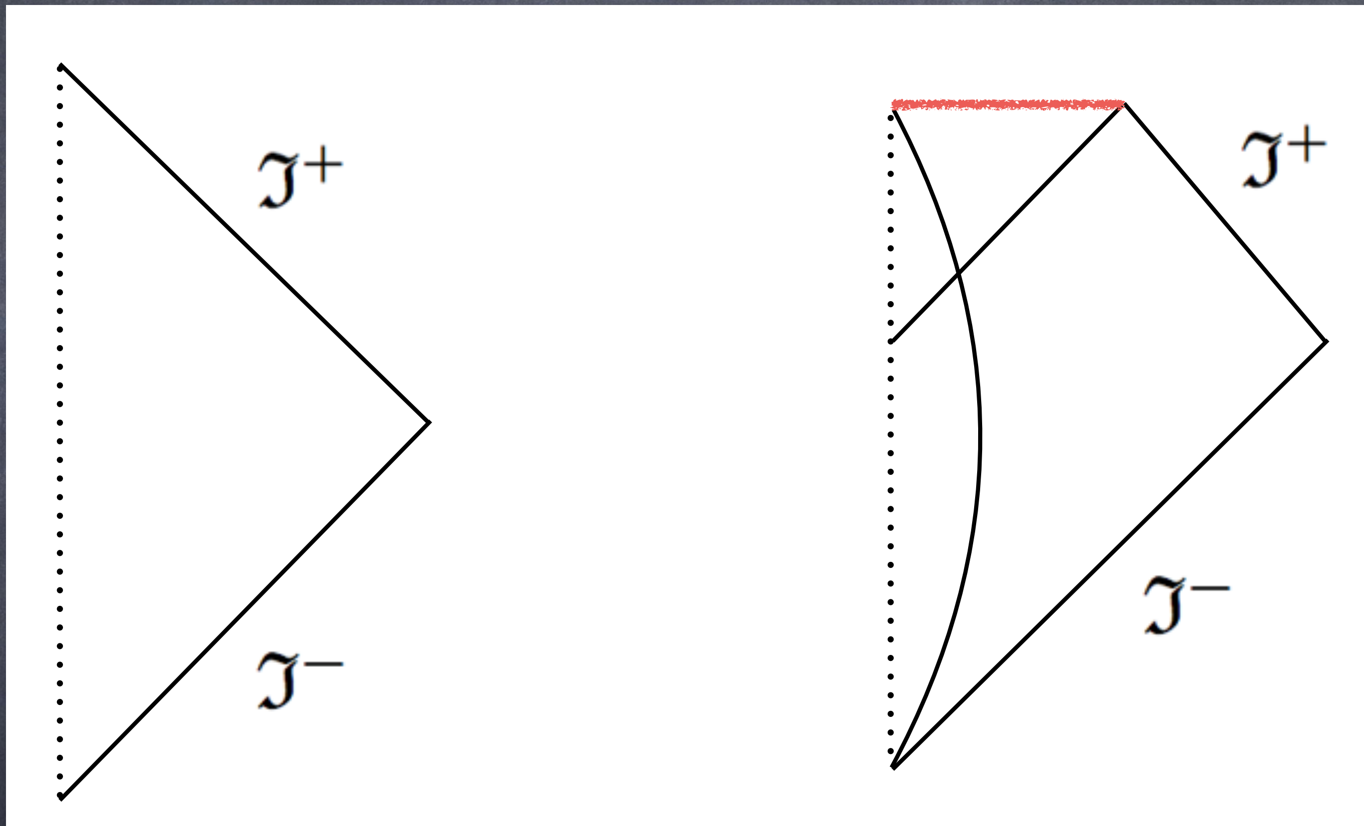
We can define the flux of momentum

$$\frac{dP^k}{dt} = -\frac{c^3}{32\pi G} r^2 \int d\Omega \langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \rangle$$

Also, the flux of angular momentum and centre of mass

Those are the flux-balance laws for Poincaré charges

Asymptotic view on flux-balance laws



Approach of Bondi, van den Burg, Metzner, Sachs (BMS), 1962

Einstein's solution

Radiative gauge [Newman-Unti gauge]

$$g_{rr} = -1, \qquad g_{ru} = g_{r\theta} = g_{r\phi} = 0$$

The asymptotic solution takes the form

$$\begin{aligned} ds^2 = &-c^2 du^2 - 2c du dr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &+ \frac{2m}{r} du^2 + r C_{AB} dx^A dx^B + \dots \\ &+ \frac{N_A}{r} du dx^A + \dots \\ &+ \frac{1}{r^i} E^{(i)}_{AB} dx^A dx^B + \dots \end{aligned}$$

It depends on

$$C_{AB}, \quad m, N_A, E^{(i)}_{AB}, \qquad i = 1, 2, \dots$$

Unconstrained Constrained

BMS flux-balance laws

Supermomentum $$Q_T(u) \equiv \int_S d^2\Omega\, m(u,\theta,\phi) T(\theta,\phi)$$

Super-Lorentz charge $$Q_R(u) \equiv \int_S d^2\Omega\, N_A(u,\theta,\phi) R^A(\theta,\phi)$$

$$Q_S^{(i)}(u) \equiv \int_S d^2\Omega\, E_{AB}^{(i)}(u,\theta,\phi) S^{AB}(\theta,\phi)$$

$$\partial_u Q_T(u) = \text{flux on } \mathfrak{I}^+ \text{dictated by Einstein's equations}$$

$$\partial_u Q_R(u) = \text{flux on } \mathfrak{I}^+ \text{dictated by Einstein's equations}$$

$$\partial_u Q_S^{(i)}(u) = \text{flux on } \mathfrak{I}^+ \text{dictated by Einstein's equations}$$

1.3. Generation of GW

Binaries as sources of GW

We consider a slowly moving source, which is weakly self-gravitating

Typical velocity → $\frac{v}{c} \ll 1$

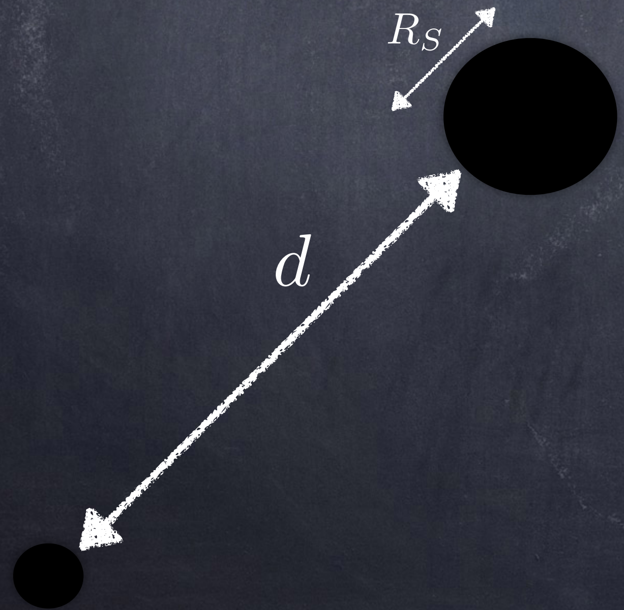
$\frac{R_S}{d} \ll 1$

Schwarzschild radius of the source: R_S

Size of the source: d

$R_S = \frac{2Gm}{c^2}$ ← Total mass

Typical example: a binary system in the early inspiral phase



Total Mass $m = m_1 + m_2$

Binaries admit only one small parameter as a result of gravitational binding

$$\frac{v}{c} \ll 1 \qquad \frac{R_S}{d} \ll 1$$

Reduced Mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Total Mass $m = m_1 + m_2$

Gravitational binding gives

$$\frac{1}{2}\mu v^2 \sim \frac{1}{2}\frac{G\mu m}{d}$$

$$\frac{v^2}{c^2} \sim \frac{R_S}{d}$$

As a result, for binary compact objects, corrections in v/c induce corrections in G.

GW wavelength in terms of the source size

- If ω_s is the typical frequency of motion inside the source and d is the typical velocities are

$$v \sim \omega_s d$$

- The frequency of radiation will also be of the order of

$$\omega_{gw} \sim 2\omega_s$$

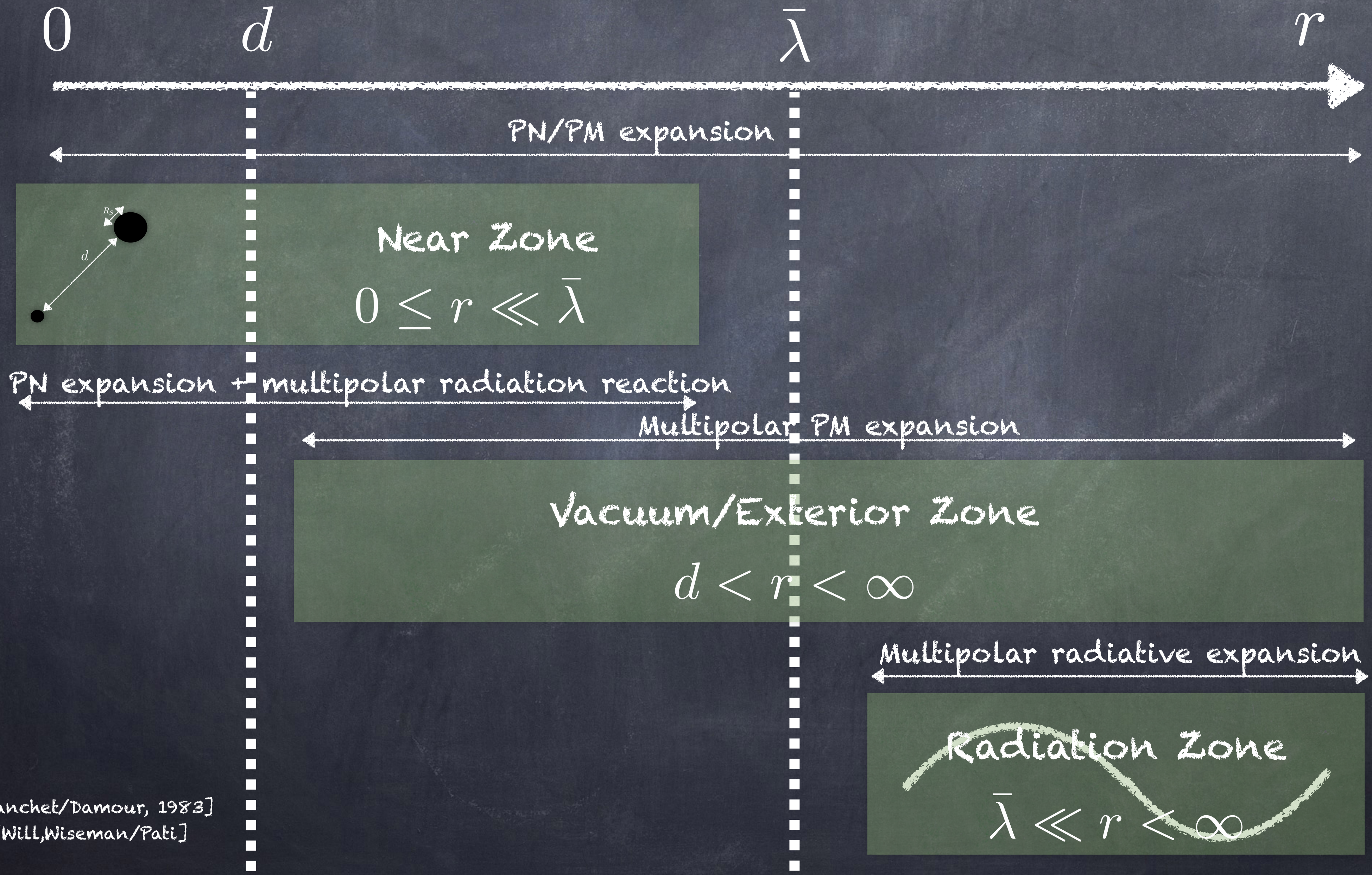
(Proof: later on !)

→ The reduced wavelength: $\bar{\lambda} = \frac{c}{\omega_{gw}} \sim \frac{c}{\omega_s} \sim \frac{c}{v} d$

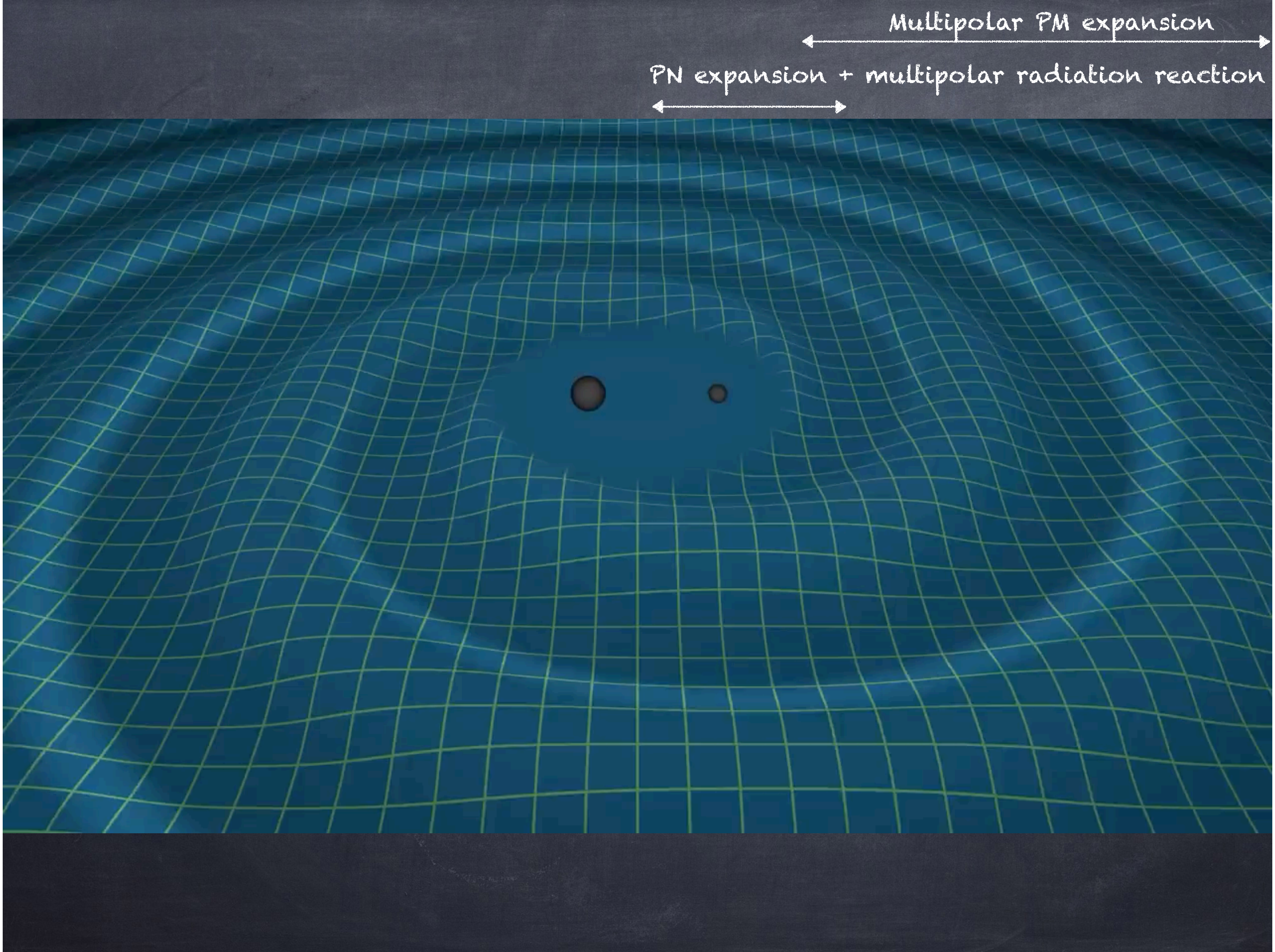
For a non-relativistic system, $v \ll c$

→ $$d \ll \bar{\lambda}$$

The 3 zones of the Post-Newtonian/ Post-Minkowskian formalism



[Blanchet/Damour, 1983]
[Will,Wiseman/Pati]


Multipolar PM expansion
PN expansion + multipolar radiation reaction

In this lecture, we will derive the motion at the lowest order:

- in the Near-Zone: Newtonian (0PN) + 2.5PN radiation-reaction
- in the exterior zone: Minkowski + linear GW (1PM)

Weak sources with arbitrary velocity

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \qquad \partial^\mu \bar{h}_{\mu\nu} = 0, \qquad \partial^\nu T_{\mu\nu} = 0$$

We can solve it in terms of the retarded Green function:

$$\Box G(x - x') = \delta^4(x - x')$$

$$\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} \int d^4x' G(x - x') T_{\mu\nu}(x')$$

Explicitly,

$$G(x - x') = \frac{-1}{4\pi|\vec{x} - \vec{x}'|} \delta(x^0_{ret} - x'^0_{ret})$$

$$x'^0 = ct' \qquad x^0_{ret} = ct_{ret} \qquad t_{ret} = t - \frac{|\vec{x} - \vec{x}'|}{c}$$

Recall that outside the source, we can use TT gauge and we have

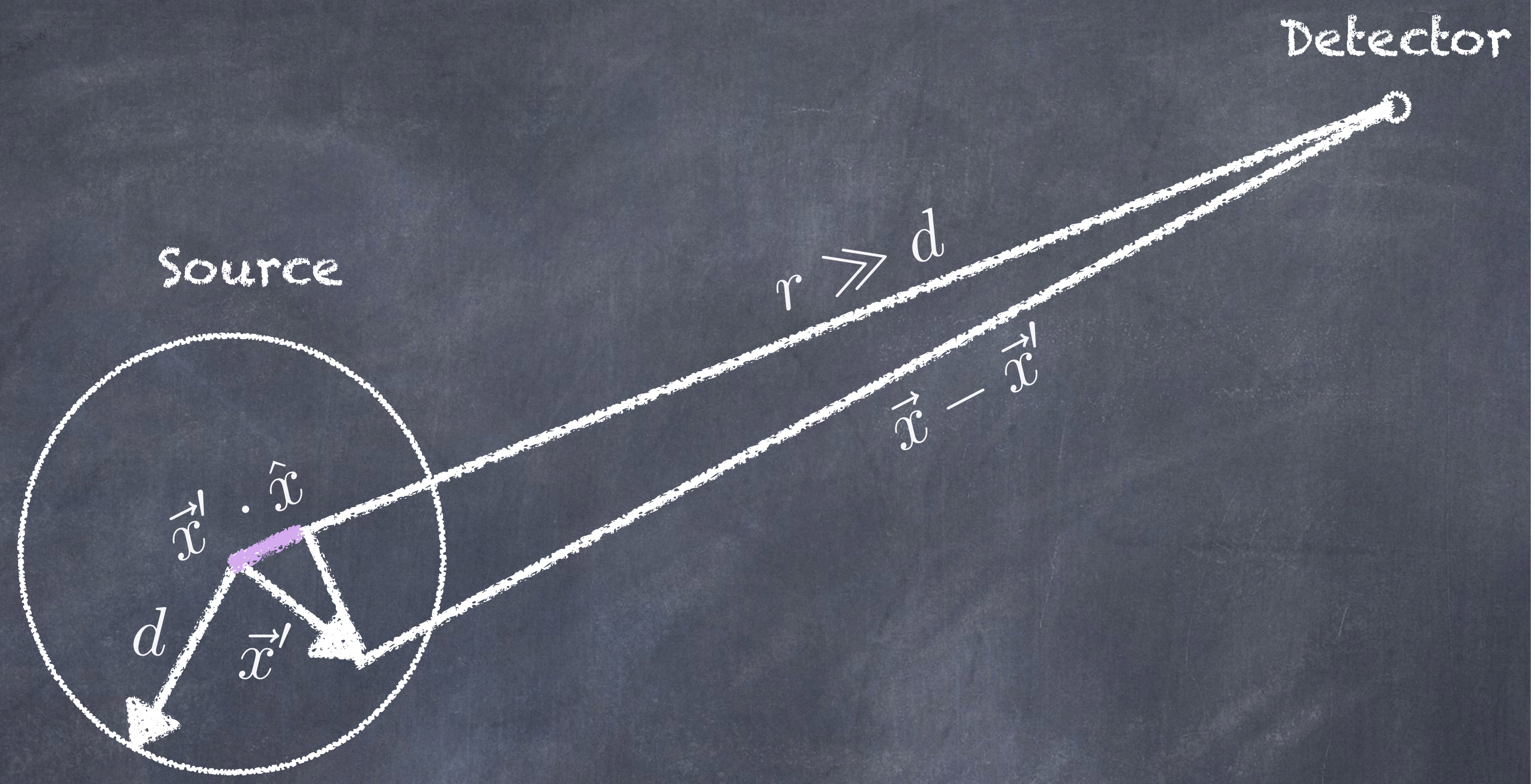
$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}$$

Therefore, outside the source we have

$$h_{ij}^{TT}(t, \vec{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\vec{x}) \int d^3x' \frac{1}{|x - x'|} T_{kl}(t - \frac{|x - x'|}{c}, x')$$

Note that h_{ij}^{TT} depends upon the integrals of the spatial components of T_{kl}

(temporal components are related by conservation)



$$|\vec{x} - \vec{x}'| = r - \vec{x}' \cdot \hat{x} + O(d^2/r), \qquad \hat{x} \equiv \frac{\vec{x}}{r}$$

Therefore, for large r,

$$h_{ij}^{TT} = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{x}) \int d^3x' T_{kl}(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c}, \vec{x}')$$

Weak sources with low velocity

Fourier transform:

$$T_{kl}\left(t-\frac{r}{c}+\frac{\vec{x}'\cdot\hat{x}}{c},\vec{x}'\right)=\int\frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\vec{k})e^{-i\omega(t-\frac{r}{c}+\frac{\vec{x}'\cdot\hat{x}}{c})+i\vec{k}\cdot\vec{x}'}$$

For a non-relativistic system,

$v \ll c$ $\qquad$ $\bar{\lambda} \gg d$

$\tilde{T}_{kl}(\omega,\vec{k})$

$\omega_s d \ll c$

and $T_{kl} \neq 0$ only inside the source $|\vec{x}'| \leq d$

$$\longrightarrow \quad \omega\frac{\vec{x}'\cdot\hat{x}}{c} \leq \frac{\omega_s d}{c} \ll 1$$

We can therefore expand

$$e^{-i\omega(t-\frac{r}{c}+\frac{\vec{x}'\cdot\hat{x}}{c})} = e^{-i\omega(t-\frac{r}{c})}[1 - i\frac{\omega}{c}\vec{x}'\cdot\hat{x} + O(\frac{\omega}{c})^2]$$

This is equivalent in position space to expanding

$$T_{kl}(t - \frac{r}{c} + \frac{\vec{x}'\cdot\hat{x}}{c}, \vec{x}') \approx T_{kl}(t - \frac{r}{c}, \vec{x}') + \frac{\vec{x}'\cdot\hat{x}}{c}\partial_t T_{kl} + O(\partial_t^2 T_{kl})$$

We define the multipoles of the stress-tensor

$$S^{ij}(t) = \int d^3x\, T^{ij}(t, \vec{x})$$

$$S^{ij,k}(t) = \int d^3x\, T^{ij}(t, \vec{x})x^k$$

$$S^{ij,kl}(t) = \int d^3x\, T^{ij}(t, \vec{x})x^k x^l$$

We get

$$h_{ij}^{TT}(t,x) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})[S^{kl} + \frac{1}{c}\hat{x}_m\dot{S}^{kl,m} + \frac{1}{2c^2}\hat{x}_m\hat{x}_p\ddot{S}^{kl,mp} + \dots]_{ret}$$

evaluated at t-r/c

$$\frac{\omega_s d}{c} \sim \frac{v}{c} \qquad O(\frac{v}{c})^2$$

Therefore, weak sources with low velocity emit gravitational radiation that is essentially determined by the lowest multipole moments

We do not need to know all the structure of the source

We only need to know its lowest multipole moments

$$h_{ij}^{TT}(t,x) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})[S^{kl} + \frac{1}{c}\hat{x}_m \dot{S}^{kl,m} + \frac{1}{2c^2}\hat{x}_m \hat{x}_p \ddot{S}^{kl,mp} + \dots]_{ret}$$

$\frac{\omega_s d}{c} \sim \frac{v}{c}$ $O(\frac{v}{c})^2$

In terms of GW power:

2.5PN 3.5PN

[will be justified later]

Two sets of multipoles have a physical interpretation:

Mass multipoles:

$$M = \frac{1}{c^2} \int d^3x T^{00}(t, x)$$

$$M^i = \frac{1}{c^2} \int d^3x T^{00}(t, x) x^i$$

...

Momentum multipoles:

$$P^i = \frac{1}{c} \int d^3x \, T^{0i}(t, x)$$

$$P^{i,j} = \frac{1}{c} \int d^3x \, T^{0i}(t, x) x^j$$

...

The multipoles of T^{ij} are related to time derivatives of these multipoles by the conservation of the stress-tensor

Exercise

Obtain the time derivative of the lowest mass and momentum multipoles

$$M^L = \frac{1}{c^2}\int d^3x T^{00} x^L \qquad P^{i,L} = \frac{1}{c}\int d^3x T^{0i} x^L$$

(i) $\dot{M} = 0$

(ii) $\dot{M}^i = P^i$

(iii) $\dot{M}^{ij} = P^{i,j} + P^{j,i}$

(iv) $\dot{P}^i = 0$

(v) $\dot{P}^{i,j} = S^{ij}$

(vi) $\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}$

using $\partial_\mu T^{\mu\nu} = 0$.

Solution

(i) $$c\dot{M} = \int d^3x \partial_0 T^{00} = -\int d^3x \partial_i T^{0i} = -\int dS_i T^{0i} = 0$$

for a volume larger than the source.

In linear theory, back reaction of GW are absent, and the mass of matter is conserved.

(ii) $$c\dot{M}^i = \int d^3x\, x^i \partial_0 T^{00} = -\int d^3x\, x^i \partial_j T^{0j}$$
$$= \int d^3x \partial_j x^i\, T^{0j} = \int d^3x\, T^{0i} = c\, P^i$$

Solution

(iii) $\dot{M}^{ij} = P^{i,j} + P^{j,i}$

$$\begin{aligned}\dot{M}_{ij} &= \frac{1}{c}\int d^3x \partial_0 T^{00} x^i x^j \\ &= -\frac{1}{c}\int d^3x \partial_k T^{0k} x^i x^j \\ &= \frac{2}{c}\int d^3x T^{0k} \partial_k x^{(i} x^{j)} \\ &= P^{i,j} + P^{j,i}\end{aligned}$$

(iv) $\dot{P}^i = 0$ This is the conservation of momentum

(v) $\dot{P}^{i,j} = S^{ij}$ [ij] -> conservation of angular momentum

(vi) $\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}$

We can combine these identities to express S^{ij}, $S^{ij,k}$ in terms of the mass and momentum multipole moments !

$$\dot{M}^{ij} = P^{i,j} + P^{j,i}$$

$$\dot{P}^{i,j} = S^{ij}$$

$$\longrightarrow \quad S^{ij} = \frac{1}{2}\ddot{M}^{ij}$$

The leading term of

$$h_{ij}^{TT}(t,x) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})[S^{kl} + \frac{1}{c}\hat{x}_m\dot{S}^{kl,m} + \frac{1}{2c^2}\hat{x}_m\hat{x}_p\ddot{S}^{kl,mp} + \ldots]_{ret}$$

becomes

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\vec{x})\ddot{M}^{kl}(t - \frac{r}{c})$$

The acceleration of mass quadrupole sources gravitational waves.

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

Radiated energy

$$\frac{dE}{dt d\Omega} = \frac{c^3 r^2}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle \qquad h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\vec{x})\ddot{M}^{kl}(t-\frac{r}{c})$$

$$= \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{n})\langle \dddot{M}_{ij}\dddot{M}_{kl}\rangle_{ret}$$

$$= \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{n})\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle_{ret} \qquad Q_{ij} = M_{ij} - \frac{1}{3}\delta_{ij}M_{kk}$$

Angular integral can be done

$$\int d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

Exercise later on

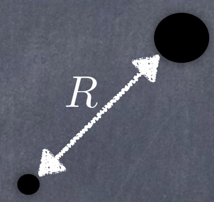
$$\longrightarrow \quad \frac{dE}{dt} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}(t-\frac{r}{c})\dddot{Q}_{ij}(t-\frac{r}{c})\rangle$$

That is the Einstein quadrupole formula

By conservation of energy,

$$\frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

For a binary source,



$$E^{source}_{0PN} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{G m_1 m_2}{R}$$

$$E^{source}_{0PN} = \frac{1}{2}m v_{c.m.}^2 + \frac{1}{2}\mu \dot{R}^2 - \frac{G m \mu}{R}$$

$$E^{source}_{0PN} = \frac{1}{2}m v_{c.m.}^2 - \frac{G m \mu}{2R}$$

Effective one-body reduction

Reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Total mass $m = m_1 + m_2$

$$\dot{R}^2 = \frac{Gm}{R}$$ Kepler law

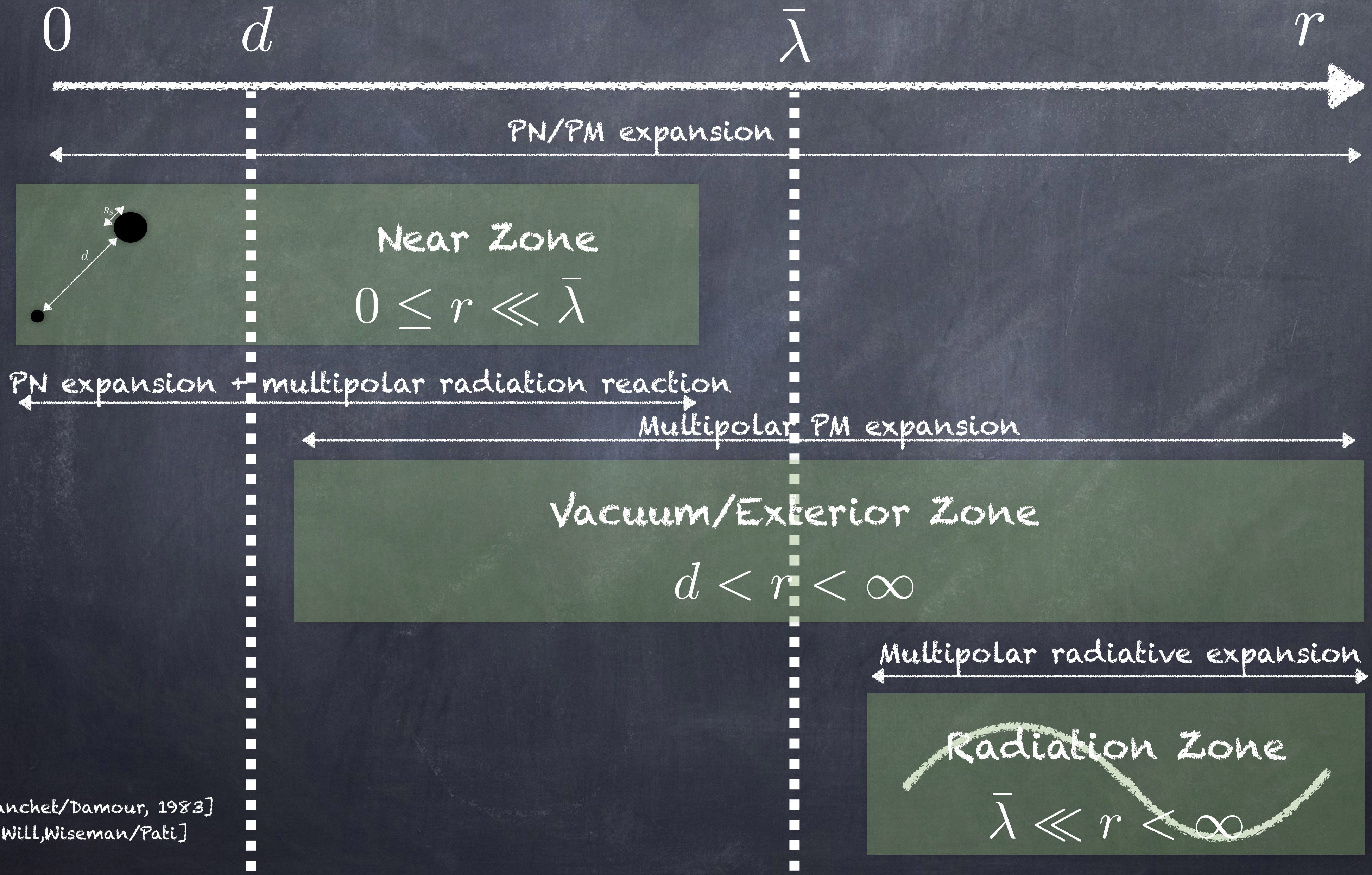
$$\frac{dE^{source}}{dt} = \frac{G m \mu}{2R^2}\dot{R}$$

$$\frac{dE}{dt} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}(t - \frac{r}{c})\dddot{Q}_{ij}(t - \frac{r}{c})\rangle$$

$$\dot{R} \sim c^{-5}$$

The gravitational wave emission brings as first contribution a 2.5PN correction to the motion. It is called the radiation reaction.

The 3 zones of the Post-Newtonian/ Post-Minkowskian formalism



[Blanchet/Damour, 1983]
[Will,Wiseman/Pati]

$$h_{ij}^{TT}(t,x) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})\left[S^{kl} + \frac{1}{c}\hat{x}_m\dot{S}^{kl,m} + \frac{1}{2c^2}\hat{x}_m\hat{x}_p\ddot{S}^{kl,mp} + \ldots\right]_{ret}$$

2.5PN 3.5PN

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}}$$
$$+ \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc]	Fokker Lagrangian
	[Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019]	Effective field theory

[review of Blanchet]

Exercise

Prove

$$\int d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

where

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

$$n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$

(ii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}|\to 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1}\cdots k^{i_\ell}}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)$

(iii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = \frac{4\pi}{\ell+1}\delta_{(i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{\ell-1} i_\ell)}$ for ℓ even

(iv) Prove $\int_S d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$

Solution

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$ $\quad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

By SO(3) symmetry, we can assume that $\vec{k}$ points in the z direction

$$\int_S d\theta d\phi \sin\theta e^{-ik_z\cos\theta} = -2\pi \int_1^{-1} d\cos\theta e^{-ik_z\cos\theta}$$
$$= \frac{2\pi}{ik_z}(e^{+ik_z} - e^{-ik_z}) = \frac{4\pi\sin k_z}{k_z}$$

Since the right-hand side is SO(3) invariant (scalar), it is af function of the norm of $\vec{k}$.

Solution

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$ $\qquad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

(ii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}|\to 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1}\cdots k^{i_\ell}}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)$ trivial

Solution

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$ $\quad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

(ii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}|\to 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1}\cdots k^{i_\ell}}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)$ trivial

(iii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = \frac{4\pi}{\ell+1}\delta_{(i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{\ell-1} i_\ell)}$

Use Taylor series

$$\frac{\sin|\vec{k}|}{|\vec{k}|} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}|\vec{k}|^{2n}$$

Only one term survives: $2n = \ell$ $\longrightarrow$ $i^\ell = (-1)^n$, $\quad \frac{\ell!}{(2n+1)!} = \frac{1}{2n+1}$

Solution

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$ $\quad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

(ii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}|\to 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1}\cdots k^{i_\ell}}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)$ trivial

(iii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = \frac{4\pi}{\ell+1}\delta_{(i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{\ell-1} i_\ell)}$

Use Taylor series

$$\frac{\sin|\vec{k}|}{|\vec{k}|} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}|\vec{k}|^{2n}$$

Only one term survives: $2n = \ell$ → $i^\ell = (-1)^n$, $\frac{\ell!}{(2n+1)!} = \frac{1}{2n+1}$

(iv) Prove $\int_S d\Omega\Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$

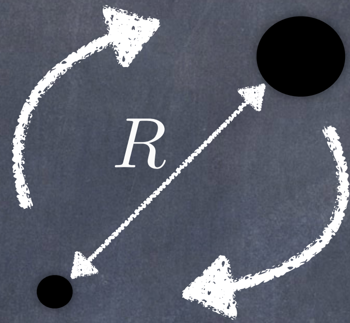
$\frac{4!}{2\cdot 2\cdot 2} = 3$ ↓

Use $\int_S d\Omega 1 = 4\pi,\quad \int_S d\Omega n_i n_j = \frac{4\pi}{3}\delta_{ij}\quad \int_S d\Omega n_i n_j n_k n_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})$

1.4. Quasi-circular inspiral of compact binaries

- Neglect conservative PN corrections (Keplerian motion + radiation reaction)
- Assume quasi-circularity

Kepler law $\omega_s^2 = \frac{Gm}{R^3}$



Total Mass $m = m_1 + m_2$

Reduced Mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Law of motion given by the flux-balance law of energy:

$$\frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

$$\frac{dE^{source}}{dt} = \frac{Gm\mu}{2R^2}\dot{R} \qquad \frac{dE}{dt} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}(t-\frac{r}{c})\dddot{Q}_{ij}(t-\frac{r}{c})\rangle$$

Exercise

Obtain the quadrupole radiation from a mass in a circular orbit

(i) Starting from the relativistic expression

$$T^{\mu\nu} = \sum_{A=1,2} \int d\tau_A m_A \frac{dx^\mu_A}{d\tau} \frac{dx^\nu_A}{d\tau} \delta^{(4)}(x - x_A(\tau_A))$$

prove that at Newtonian order:

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

(ii) Deduce the quadrupole formula

$$Q^{ij}(t) = \left[\frac{1}{c^2} \int d^3x T^{00}(t, \vec{x}) x^i x^j\right]^{STF} = \mu \left(x_0^i(t) x_0^j(t) - \frac{1}{3} r_0^2(t) \delta^{ij} \right)$$

where $\vec{x}_0 \equiv \vec{x}_1 - \vec{x}_2$

Solution

(i)

$$T^{\mu\nu} = \sum_{A=1,2} \int d\tau_A m_A \frac{dx^\mu_A}{d\tau_A} \frac{dx^\nu_A}{d\tau_A} \delta^{(4)}(x - x_A(\tau_A))$$

$$c^2 d\tau_A^2 = -\eta_{\mu\nu} dx^\mu_A dx^\nu_A = c^2 (1 - \frac{v_A^2}{c^2}) dt^2 = c^2 \gamma_A^{-2} dt^2 \longrightarrow d\tau_A = \frac{dt}{\gamma_A}$$

$$T^{\mu\nu} = \sum_{A=1,2} \gamma_A m_A \frac{dx^\mu_A}{dt} \frac{dx^\nu_A}{dt} \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} \gamma_A m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

Newtonian limit:

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

(ii)

$$T^{00}(t,\vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x}-\vec{x}_A(t))$$

$$M^{ij} = \frac{1}{c^2}\int d^3x T^{00}(t,\vec{x}) x^i x^j$$

$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m x_{c.m.}^i x_{c.m.}^j + \mu x_0^i x_0^j$$

$$\vec{x}_0 \equiv \vec{x}_1 - \vec{x}_2 \qquad \vec{x}_{c.m.} \equiv \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1+m_2} = 0$$

$$Q^{ij}(t) = \left[\frac{1}{c^2}\int d^3x T^{00}(t,\vec{x}) x^i x^j\right]^{STF}$$

$$Q^{ij}(t) = \mu\left(x_0^i(t) x_0^j(t) - \frac{1}{3} r_0^2(t)\delta^{ij}\right)$$

Explicit expression

Assume orbital motion is exactly circular (not elliptic)

$$z = 0$$

$$\begin{cases} x_0(t) = R\cos(\omega_s t + \frac{\pi}{2}) \\ y_0(t) = R\sin(\omega_s t + \frac{\pi}{2}) \\ z_0(t) = 0 \end{cases}$$

$$M^{ij} = \mu x_0^i(t) x_0^j(t)$$

$$M^{11} = \mu R^2 \cos^2(\omega_s t + \frac{\pi}{2})$$

$$M^{12} = \mu R^2 \sin(\omega_s t + \frac{\pi}{2}) \cos(\omega_s t + \frac{\pi}{2})$$

$$M^{22} = \mu R^2 \sin^2(\omega_s t + \frac{\pi}{2})$$

Metric perturbation

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\vec{x})\ddot{M}^{kl}(t-\frac{r}{c})$$

For a generic direction $\hat{n} = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$

we find

$$\begin{aligned} h_+(t,\theta,\phi) = \frac{1}{r}\frac{G}{c^4}[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta)] \\ + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) \\ - \ddot{M}_{33}\sin^2\theta \\ - \ddot{M}_{12}\sin 2\phi(1+\cos^2\theta) \\ + \ddot{M}_{13}\sin\phi\sin 2\theta \\ + \ddot{M}_{23}\cos\phi\sin 2\theta] \end{aligned}$$

$$\begin{aligned} h_\times(t,\theta,\phi) = \frac{1}{r}\frac{G}{c^4}[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta] \\ + 2\ddot{M}_{12}\cos 2\phi\cos\theta \\ - 2\ddot{M}_{13}\cos\phi\sin\theta \\ + 2\ddot{M}_{23}\sin\phi\sin\theta] \end{aligned}$$

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}(\frac{1+\cos^2\theta}{2})\cos(2\omega_s t_{ret}+2\phi)$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos\theta\sin(2\omega_s t_{ret}+2\phi)$$

Quadrupole radiation is at twice the frequency of the source.

$\omega_{gw} \sim 2\omega_s$ [as announced earlier]

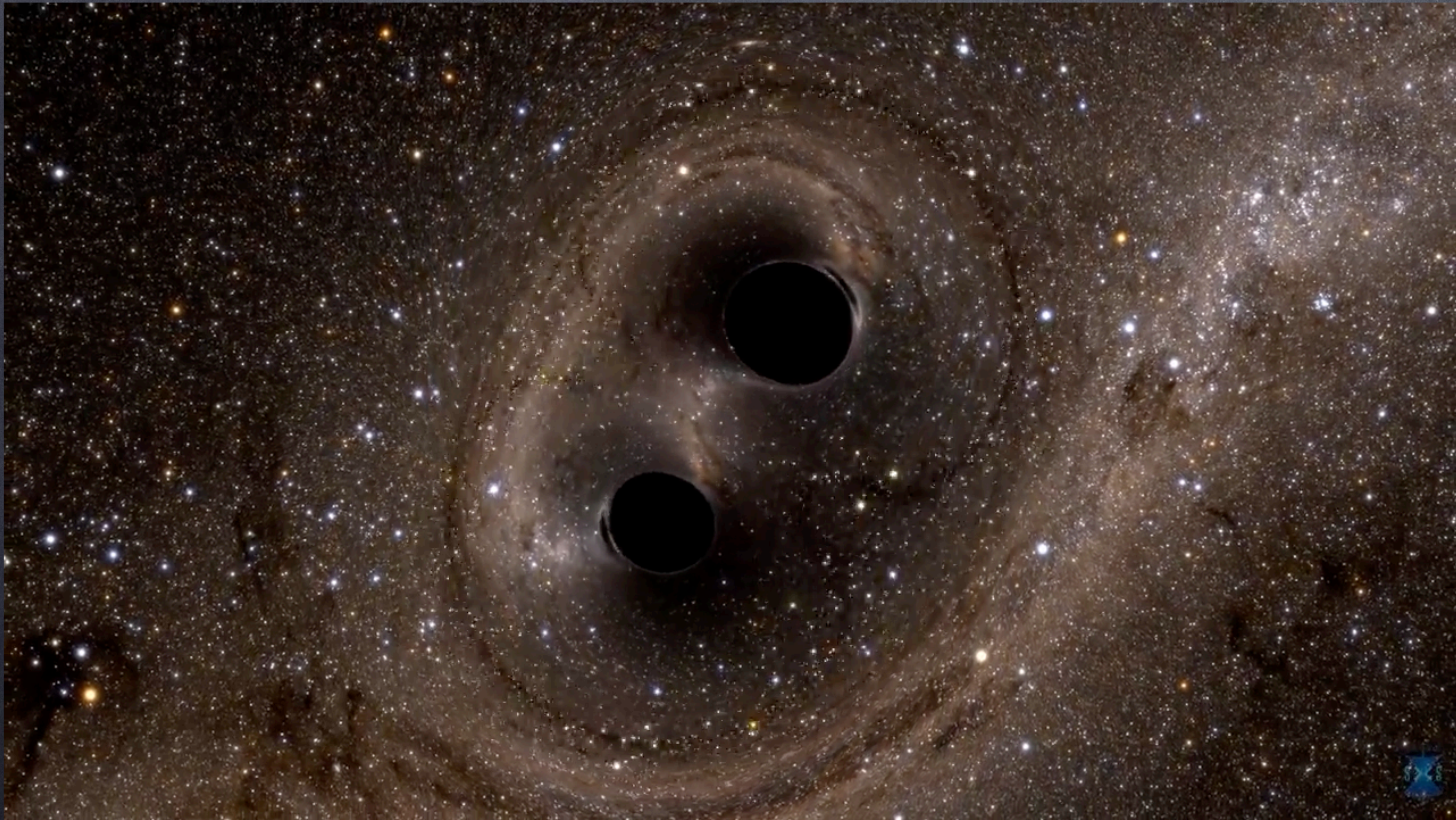
Helicoidal structure

Performing a rotation is equivalent to shifting time

The angle θ is the angle between the normal to the orbit and the line-of-sight

The distance r is a constant

Face-on $\theta = 0$



Edge-on $\theta = \frac{\pi}{2}$

Compute the angular distribution of radiated power in the quadrupole approximation

$$\frac{dE}{dtd\Omega} = \frac{r^2 c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

using

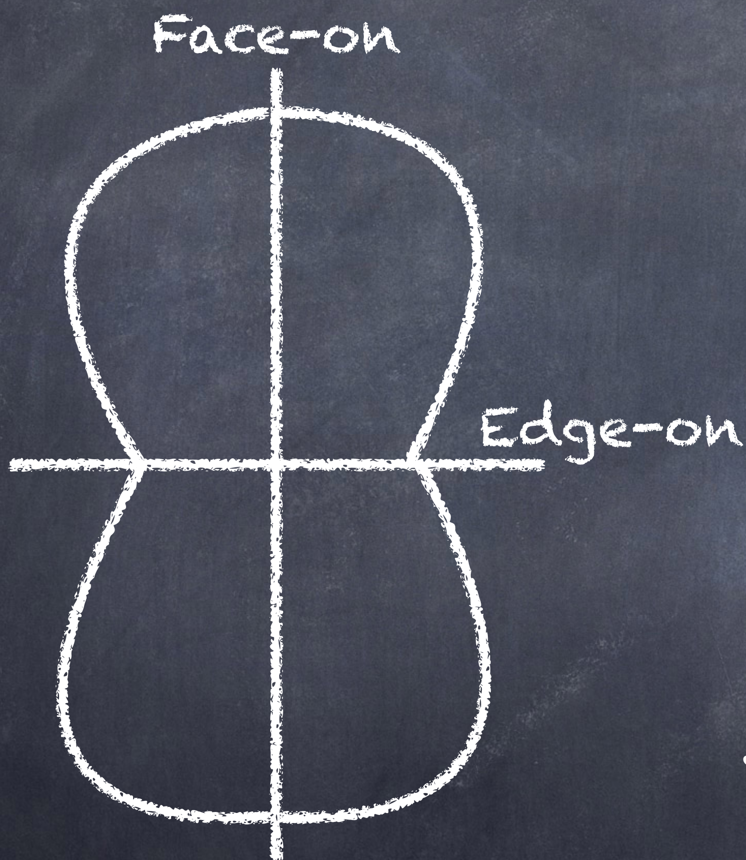
$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}(\frac{1+\cos^2\theta}{2})\cos(2\omega_s t_{ret} + 2\phi)$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\theta\sin(2\omega_s t_{ret} + 2\phi)$$

Use

$$\langle\cos^2(2\omega_s t_{ret} + \phi)\rangle = \langle\sin^2(2\omega_s t_{ret} + \phi)\rangle = \frac{1}{2}$$

$$\frac{dE}{dt d\Omega} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$



$$g(\theta) = (\frac{1+\cos^2\theta}{2})^2 + \cos^2\theta$$

$$\int \frac{d\Omega}{4\pi} g(\theta) = \frac{4}{5}$$

$$P_{rad} = \frac{dE}{dt} = \frac{32}{5}\frac{G\mu^2}{c^5} R^4 \omega_s^6 = \frac{32}{5}\frac{G\mu^2}{R^2}(\frac{v}{c})^6$$

Inspiral motion to coalescence as a consequence of energy flux-balance

Emission of GW costs energy, taken from the system's total energy

$$\frac{dE^{source}}{dt} = \frac{Gm\mu}{2R^2}\dot{R} \qquad \frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

⟶ R decreases with time

⟶ $\omega_s^2 = \frac{Gm}{R^3}$ increases with time

⟶ power radiated in GW increases

⟶ Runaway process leading to coalescence

Exercise The motion is quasi-circular as long as $\dot{R} \ll v$

Prove that it is equivalent to $\frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1$

Solution

$$R = \left(\frac{Gm}{\omega_s^2}\right)^{1/3}$$ Kepler's law

$$\begin{aligned}\dot{R} &= -\frac{2}{3}R\frac{\dot{\omega}_s}{\omega_s} \\ &= -\frac{2}{3}(\omega_s R)\frac{\dot{\omega}_s}{\omega_s^2} \\ &= -\frac{2}{3}v\frac{\dot{\omega}_s}{\omega_s^2}\end{aligned}$$

$$\dot{R} \ll v \Leftrightarrow \frac{\dot{\omega}_s}{\omega_s^2} \ll 1 \Leftrightarrow \frac{\dot{\omega}_{gw}}{\omega_{gw}^2} \ll 1 \Leftrightarrow \frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1$$

Exercise

Rewrite

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}(\frac{1+\cos^2\theta}{2})\cos(2\omega_s t_{ret} + 2\phi)$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\theta\sin(2\omega_s t_{ret} + 2\phi)$$

using

$$R = (\frac{Gm}{\omega_s^2})^{1/3} \qquad \omega_s = \frac{1}{2}\omega_{gw} = \frac{1}{2}2\pi f_{gw} = \pi f_{gw}$$

and the chirp mass

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

Keep M_c , m and f_{gw} . What do you notice?

Solution

$$h_+(t) = \frac{4}{r}(\frac{GM_c}{c^2})^{5/3}(\frac{\pi f_{gw}}{c})^{2/3}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{gw}t_{ret} + 2\phi)$$

$$h_\times(t) = \frac{4}{r}(\frac{GM_c}{c^2})^{5/3}(\frac{\pi f_{gw}}{c})^{2/3}\cos\theta\sin(2\pi f_{gw}t_{ret} + 2\phi)$$

All dependence in the mass is through the chirp mass at this lowest PN order!

Power radiated:

$$\frac{dP}{d\Omega} = \frac{2}{\pi}\frac{c^5}{G}(\frac{GM_c\omega_{gw}}{2c^3})^{10/3}g(\theta)$$

$$P = \frac{32}{5}\frac{c^5}{G}(\frac{GM_c\omega_{gw}}{2c^3})^{10/3}$$

Exercise

(i) Prove

$$E^{source} = -\frac{G\mu m}{2R} = -\left(\frac{G^2 M_c^5 \omega_{gw}^2}{32}\right)^{1/3}$$

using

$$\omega_{gw} = 2\omega_s \qquad R = (\frac{Gm}{\omega_s^2})^{1/3} \qquad M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

(ii) Derive the evolution of the GW frequency $f_{gw} = \frac{\omega_{gw}}{2\pi}$

using the energy flux-balance law with GW power

$$P = \frac{32}{5}\frac{c^5}{G}(\frac{GM_c\omega_{gw}}{2c^3})^{10/3}$$

Solution

$$\dot{f}_{gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{gw}^{11/3}$$

It integrates to

$$f_{gw}(t) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{t_{coal}-t}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

time of coalescence

Validity of the quasi-circularity hypothesis:

$$\frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1 \Leftrightarrow \left(\frac{96}{5}\pi^{8/3}\right)^{3/5}\frac{GM_c}{c^3} f_{gw} \ll 1$$

To get the waveform, we need the orbital motion with 2.5PN radiation-reaction (Matching between Near Zone and Exterior zone)

$z = 0$

$$\begin{cases} x_0(t) = R\cos(\omega_s t + \frac{\pi}{2}) \longrightarrow R(t)\cos\frac{\Phi(t)}{2} \\ y_0(t) = R\sin(\omega_s t + \frac{\pi}{2}) \longrightarrow R(t)\sin\frac{\Phi(t)}{2} \\ z_0(t) = 0 \end{cases}$$

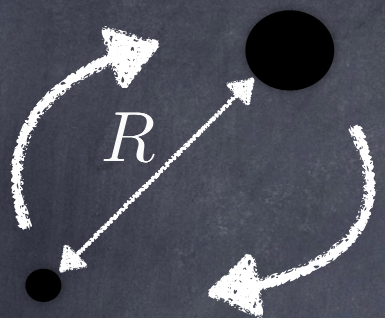
where

$$\Phi(t) = \int_{t_0}^{t} dt' \omega_{gw}(t') = 2\int_{t_0}^{t} dt' \omega_s(t')$$

$$\Phi(t) = \Phi_0 - 2\left(\frac{5GM_c}{c^3}\right)^{-5/8} (t_{coal} - t)^{5/8}$$

-> Quadupole moment/Waveform gets corrections.

Quadrupole moment with radiation-rection



$$M^{ij} = \mu x_0^i(t) x_0^j(t)$$

$$M^{11} = \mu R^2 \cos^2(\omega_s t + \frac{\pi}{2})$$

$$M^{12} = \mu R^2 \sin(\omega_s t + \frac{\pi}{2}) \cos(\omega_s t + \frac{\pi}{2})$$

$$M^{22} = \mu R^2 \sin^2(\omega_s t + \frac{\pi}{2})$$

Replace

$$R \to R(t)$$

$$\omega_s t + \frac{\pi}{2} \to \Phi(t)/2$$

However, $\dot{R}$ is negligible as long as the orbit is quasi-circular. Therefore, we can ignore these terms.

Therefore, the waveform $h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl}(\vec{x}) \ddot{M}^{kl}(t - \frac{r}{c})$

is simply obtained by making the replacement $\omega_s t + \frac{\pi}{2} \to \Phi(t)/2$

Final 0PN+2.5PN rad-react/1PM GW waveform

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t_{\cancel{ret}})}{c}\right)^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos\Phi(t_{\cancel{ret}})$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t_{\cancel{ret}})}{c}\right)^{2/3}(\cos\theta)\sin\Phi(t_{\cancel{ret}})$$

where

$$\Phi(t) = \Phi_0 - 2\left(\frac{5GM_c}{c^3}\right)^{-5/8}(t_{coal}-t)^{5/8}$$

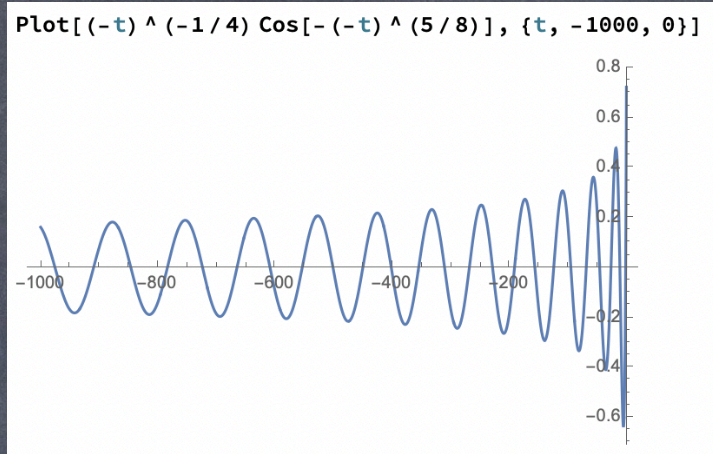
$$f_{gw}(t) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{t_{coal}-t}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

The retardation is a fixed time shift. Instead we express the waveform as a function of time before coalescence.

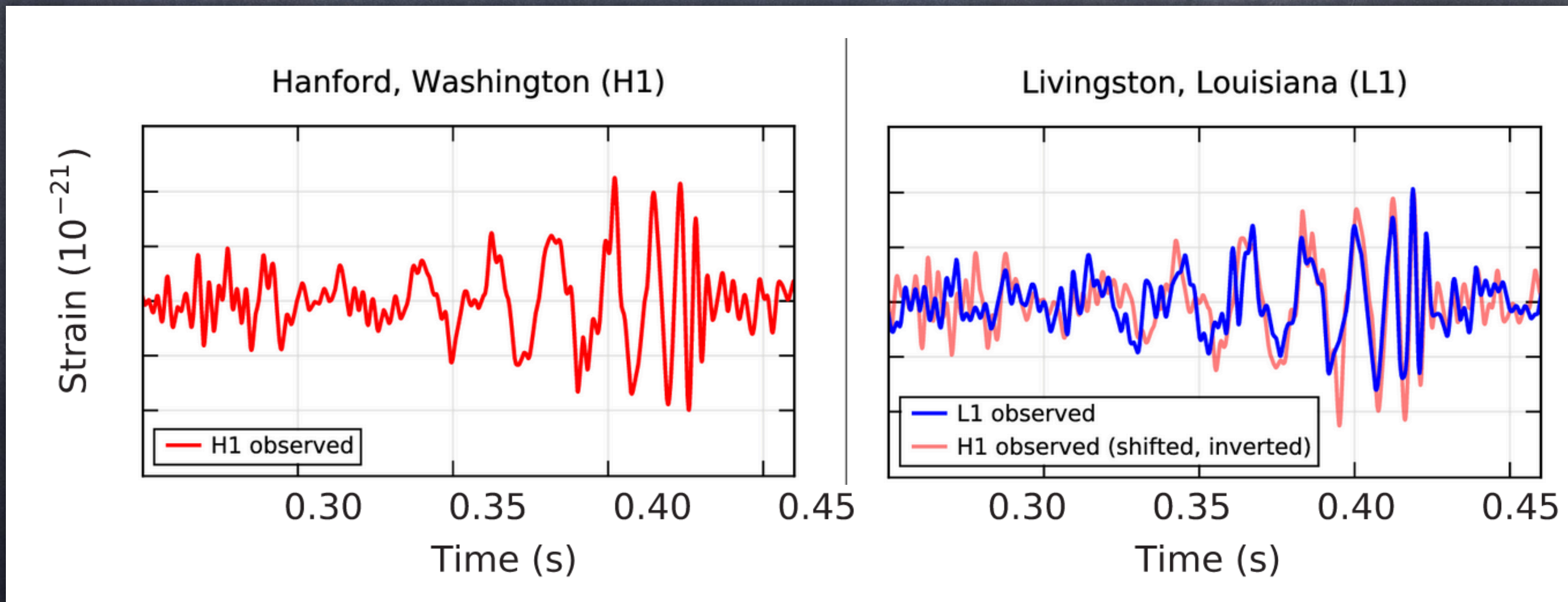
$$h_+(t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c(t_{coal}-t)}\right)^{1/4}\left(\frac{1+\cos^2\theta}{2}\right)\cos\Phi(t_{coal}-t)$$

$$h_\times(t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c(t_{coal}-t)}\right)^{1/4}\cos\theta\sin\Phi(t_{coal}-t)$$

Compare 0PN+2.5PN rad-react/1PM GW waveform

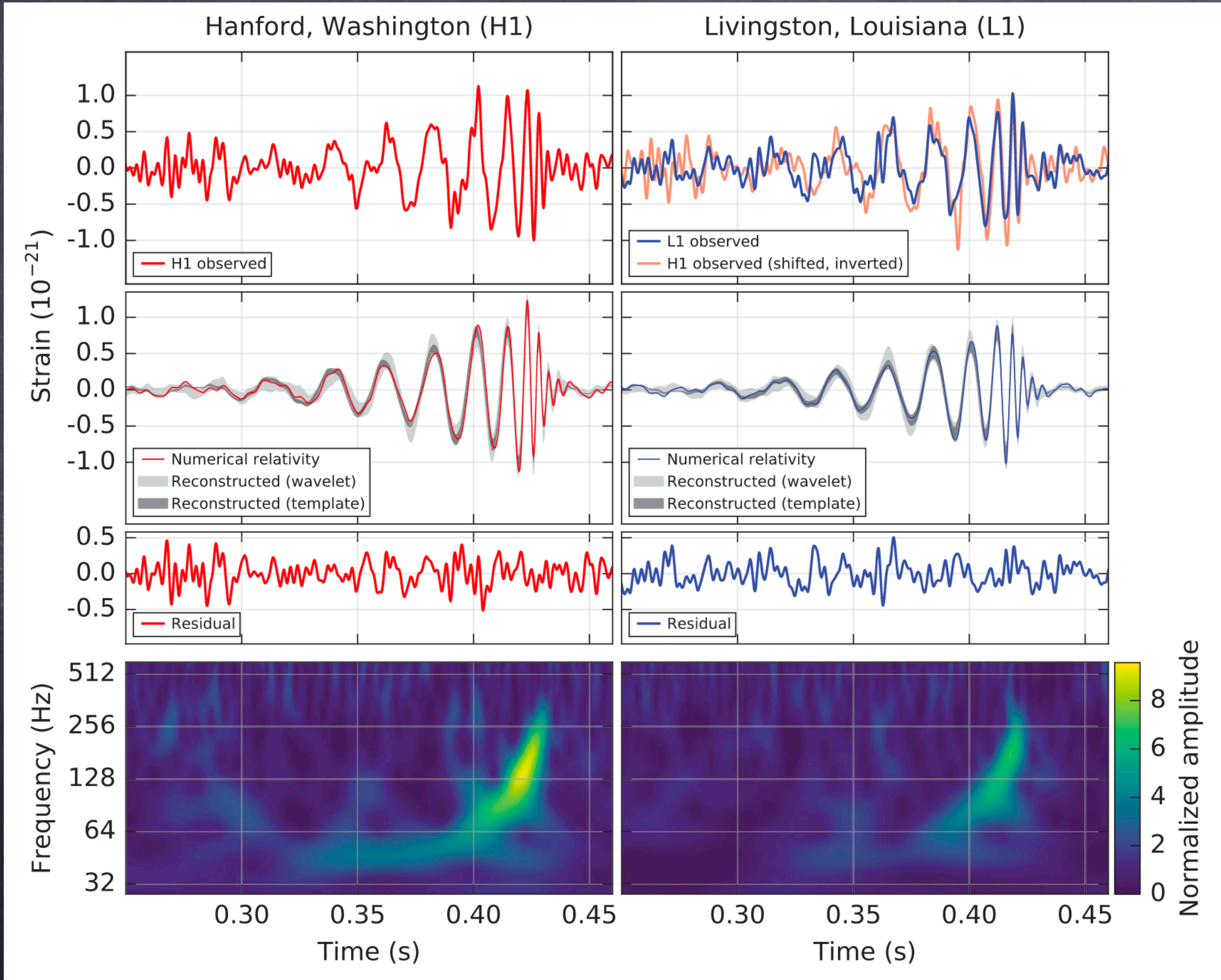


with GW150914



What do you observe?

Actual GW template used



Exercise

Match with GW150914

Estimated source parameters

Quantity	Value	Upper/Lower error estimate	Unit
Primary black hole mass	**36.2**	+5.2 -3.8	M sun
Secondary black hole mass	**29.1**	+3.7 -4.4	M sun
Final black hole mass	**62.3**	+3.7 -3.1	M sun
Final black hole spin	**0.68**	+0.05 -0.06	
Luminosity distance	**420**	+150 -180	Mpc
Source redshift, *z*	**0.09**	+0.03 -0.04	
Energy radiated	**3.0**	+0.5 -0.5	M sun

$$\frac{2GM_\odot}{c^2} = 3\ \mathrm{km}$$

$$1\ \mathrm{pc} = 3.3\ \text{light-years}$$

$$1\ \text{light-year} = 9.5 \times 10^{15} m$$

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

(i) Obtain the GW amplitude 0.2 sec before merger

$$A(t_{coal} - t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c(t_{coal} - t)}\right)^{1/4}$$

(ii) Obtain the GW frequency 0.2 sec before merger

$$f_{gw}(t_{coal} - t) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{t_{coal} - t}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

Solution

(i)

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \longrightarrow M_c = 28 M_\circ$$

$$\frac{2GM_\circ}{c^2} = 3 \text{ km} \longrightarrow \frac{GM_c}{c^2} = 42 \text{ km}$$

$$420 \text{ Mpc} = 1.3 \times 10^{22} \text{ km} \qquad \frac{(42)^{5/4} 5^{1/4}}{1.3 \times 10^{22} (3 \times 10^5)^{1/4} (0.2)^{1/4}} = 7.8 \times 10^{-22}$$

$$\mathcal{A}(t) = 7.8 \times 10^{-22} \left(\frac{420 \text{ Mpc}}{r} \right) \left(\frac{M_c}{28 M_\circ} \right)^{5/4} \left(\frac{0.2 \text{ sec}}{t_{coal} - t} \right)^{1/4}$$

(ii)

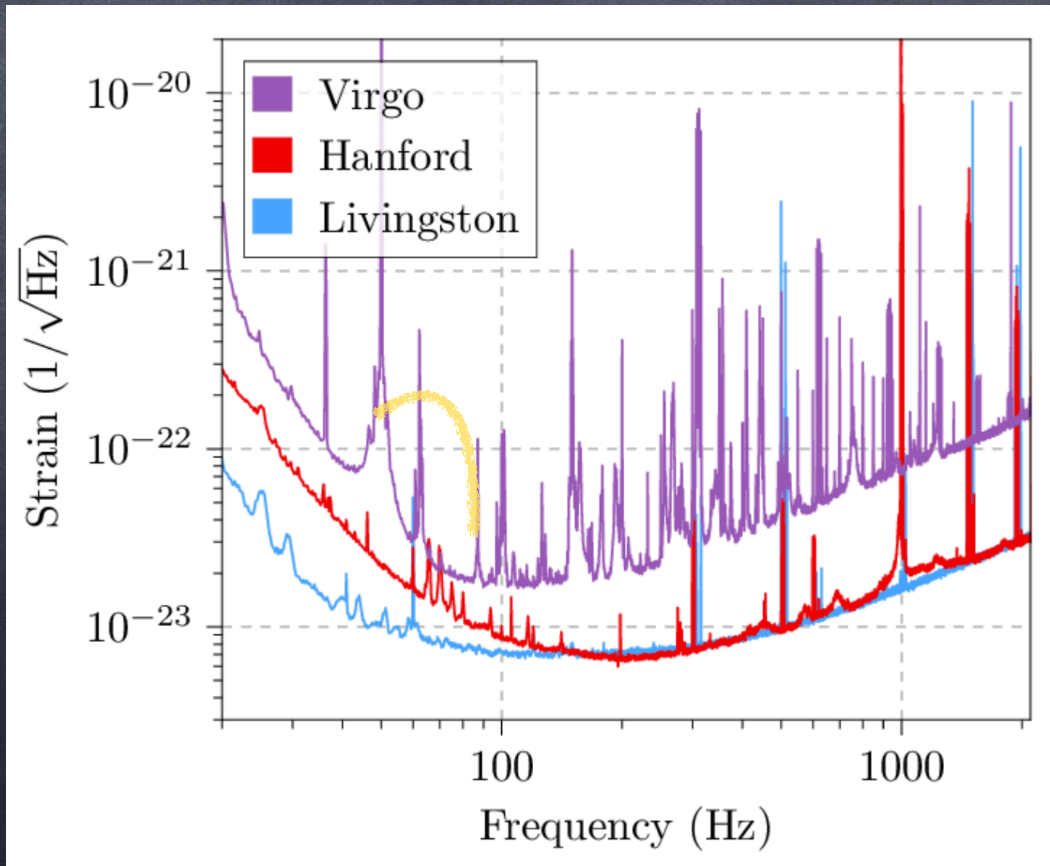
$$\frac{1}{\pi} \left(\frac{5}{256} \right)^{3/8} \frac{42^{-5/8} (3 \times 10^5)^{5/8}}{0.2^{3/8}} = 34$$

$$f_{gw}(t) = 34 \, \text{Hz} \left(\frac{M_c}{28 M_\circ} \right)^{-5/8} \left(\frac{0.2 \text{ sec}}{t_{coal} - t} \right)^{3/8}$$

Model versus LIGO sensitivity

$$\mathcal{A}(t) = 7.8 \times 10^{-22} \left(\frac{420\ \text{Mpc}}{r}\right) \left(\frac{M_c}{28 M_\circ}\right)^{5/4} \left(\frac{0.2\ \text{sec}}{t_{coal} - t}\right)^{1/4}$$

$$f_{gw}(t) = 34\,\text{Hz} \left(\frac{M_c}{28 M_\circ}\right)^{-5/8} \left(\frac{0.2\ \text{sec}}{t_{coal} - t}\right)^{3/8}$$



Challenges within the PN/PM formalism

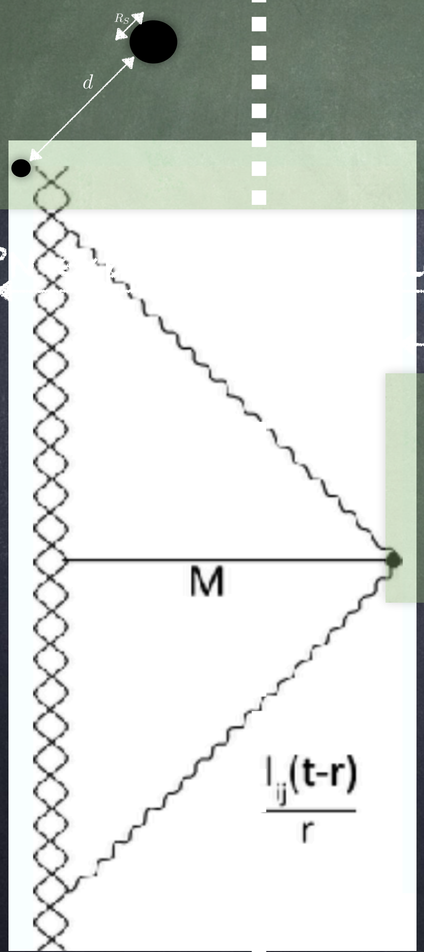
[Blanchet/Damour, 1988]

0 — d — $\bar{\lambda}$ — r

PN/PM expansion

Near Zone

$0 \leq r \ll \bar{\lambda}$



P… …tipolar radiation reaction

Multipolar PM expansion

Vacuum/Exterior Zone

$d < r < \infty$

There is a "hereditary" PM2/PN4 term in the equation of motion of the source.
It is non-local in time.
The mathematics of such evolution equations is not yet fully understood.

Limit of the PN/PM formalism

The PN/PM formalism leads to a perturbation series around a Keplerian orbit. After taking into account the backreaction of the central black hole, relativistic effects come in.

Schwarzschild admits an innermost stable circular orbit (ISCO)

$$r_{ISCO} = \frac{6Gm}{c^2}$$

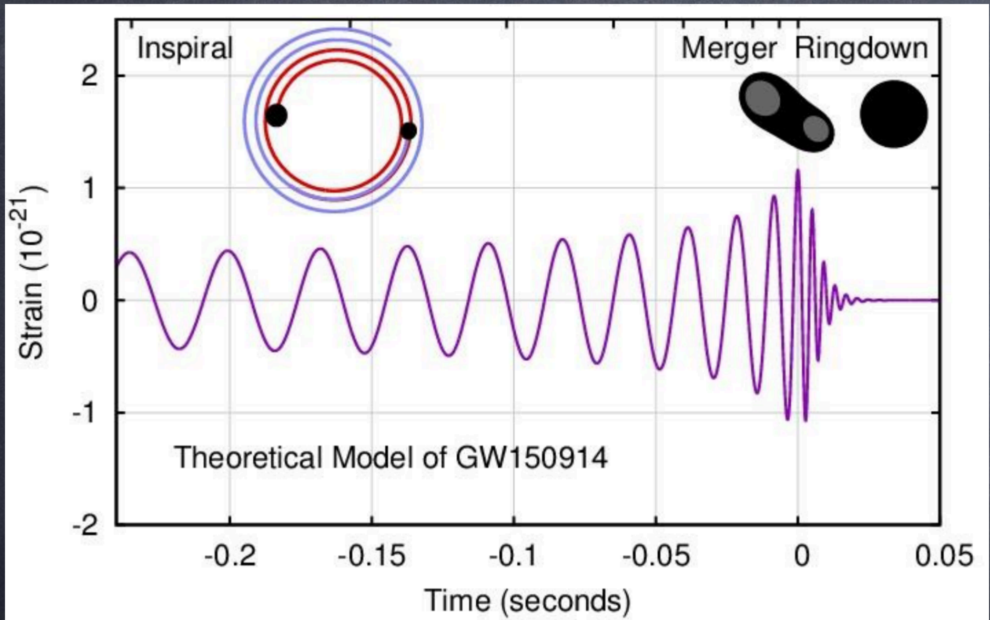
The quasi-circularity breaks down because there are no more stable circular orbits beyond that radius. This regime is called the transition to merger. The maximal frequency where the PN/PM formalism is valid can be approximated to

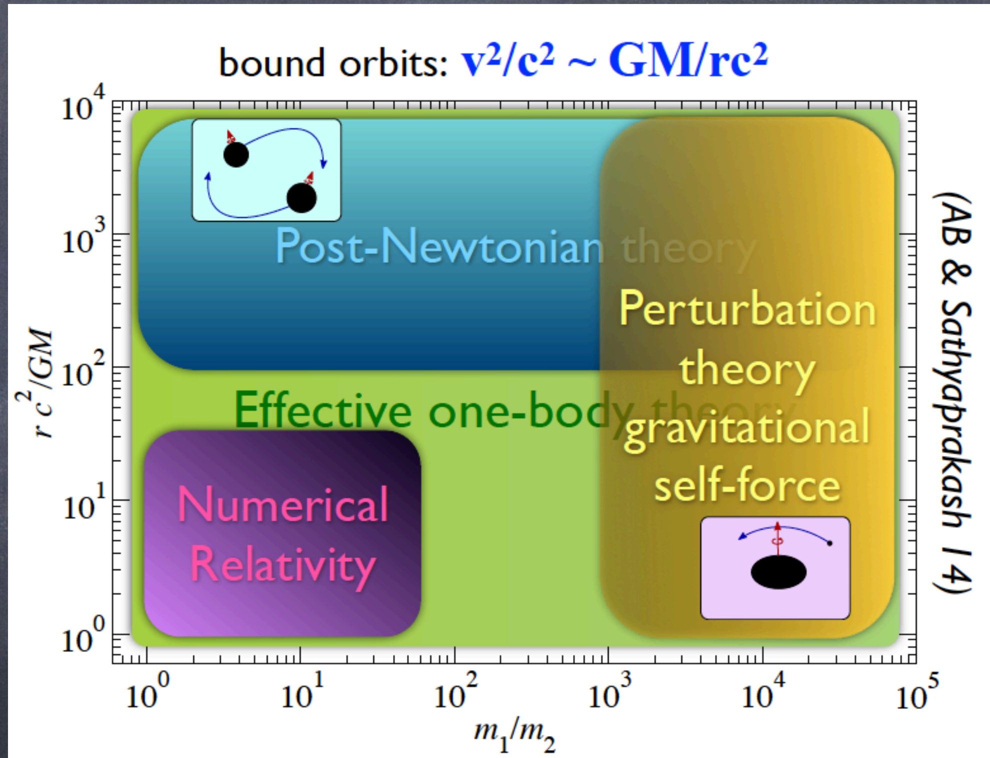
$$f_{gw}^{MAX} = 2f_s = \frac{2\omega_s}{2\pi} = \frac{1}{\pi}\sqrt{\frac{Gm}{R^3}} = \frac{1}{6\sqrt{6}\pi}\frac{c^3}{Gm} \sim 34\ \mathrm{Hz}\frac{65M_\circ}{m}$$

Note that it occurs before the end of the quasi-circularity hypothesis computed earlier:

$$f_{gw}^{MAX,QC} = \left(\frac{96}{5}\pi^{8/3}\right)^{-3/5}\frac{c^3}{GM_c} \sim 194\mathrm{Hz}\frac{28M_\circ}{M_c}$$

Beyond PN/PM Formalism





(AB & Sathyaprakash 14)