ADVANCED QUANTUM FIELD THEORY

Exercises

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1 Exercises: functional integral methods

1.1 Linear operators on $L^2(\mathbf{R}^4)$

Quite often one has to manipulate linear operators $\widehat{A}$ acting on functions $f \in L^2(\mathbf{R}^4)$. These are *not* operators acting in the Hilbert space of quantum field theory but in a "much smaller" Hilbert space. A typical example are differential operators. In general one can write

$$(\widehat{A}f)(x) = \int \mathrm{d}^4 y \, A(x,y) f(y) \ . \tag{1.1}$$

It is very convenient to use the notation $A(x,y) = \langle x | \widehat{A} | y \rangle$ where $|x\rangle$ is the ket of the non-normalisable basis of $L^2(\mathbf{R}^4)$ such that $\widehat{X}^\mu |x\rangle = x^\mu |x\rangle$ (eigenstate of the position operator). One also uses the non-normalisable eigenstates $|p\rangle$ of $\widehat{P}^\mu$. Recall that

$$\mathbf{1} = \int \mathrm{d}^4 x \ |x\rangle \langle x| = \int \mathrm{d}^4 p \, |p\rangle \langle p| \quad , \quad \langle x | \, p \rangle = \frac{e^{ipx}}{(2\pi)^2} \ . \tag{1.2}$$

a) Show that an nth order differential operator such that $(\widehat{A}f)(x) = \frac{\partial}{\partial x^{\mu_1}} \ldots \frac{\partial}{\partial x^{\mu_n}} f(x)$, corresponds to $A(x,y) = (-)^n \frac{\partial}{\partial y^{\mu_1}} \ldots \frac{\partial}{\partial y^{\mu_n}} \delta^{(4)}(x-y) = \frac{\partial}{\partial x^{\mu_1}} \ldots \frac{\partial}{\partial x^{\mu_n}} \delta^{(4)}(x-y)$. Show that for this same operator one also has $A(p,\widetilde{p}) \equiv \langle p | \widehat{A} | \widetilde{p} \rangle = i^n \, p_{\mu_1} \ldots p_{\mu_n} \delta^{(4)}(p - \widetilde{p})$.

b) Show that for any $\widehat{A}$, the matrix elements $A^{-1}(x,y) \equiv \langle x | \widehat{A}^{-1} | y \rangle$ of the inverse operator satisfy

$$\int \mathrm{d}^4 z \, A^{-1}(x,z) \, A(z,y) = \delta^{(4)}(x-y) \ . \tag{1.3}$$

Write the analogous relation for $A^{-1}(p,\widetilde{p}) = \langle p | \widehat{A}^{-1} | \widetilde{p} \rangle$ and $A(\widetilde{p}, p')$.

c) Show that if $A(x,y)$ only depends on the difference $x-y$ (translation invariance), then

$$\langle p | \widehat{A} | \widetilde{p} \rangle = A(p) \, \delta^{(4)}(p - \widetilde{p}) \quad \text{with} \quad A(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, e^{ip(x-y)} \, A(p) \ , \tag{1.4}$$

and that in this case the matrix elements of the inverse operator simply are

$$\langle p | \widehat{A}^{-1} | \widetilde{p} \rangle = \frac{1}{A(p)} \, \delta^{(4)}(p - \widetilde{p}) \quad \Rightarrow \quad A^{-1}(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, e^{ip(x-y)} \, \frac{1}{A(p)} \ . \tag{1.5}$$

More generally, show that for such operators one has for a "general" function F of $\widehat{A}$:

$$\langle x | \, F(\widehat{A}) \, | y \rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, e^{ip(x-y)} F(A(p)) \ . \tag{1.6}$$

d) Use these results to show that the inverse of the Feynman propagator of a scalar field $A(x,y) = \Delta_F(x-y)$ is given by $A^{-1}(x,y) = \Big[-\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} + m^2 - i\epsilon \Big] \delta^{(4)}(x-y)$. Deduce the differential equation satisfied by the Feynman propagator.

f) We will also have to compute traces of certain operators on $L^2(\mathbf{R}^4)$. By definition

$$\mathrm{Tr}\,\widehat{A} = \int \mathrm{d}^4x\ \langle x|\,\widehat{A}\,|x\rangle = \int \mathrm{d}^4x\, A(x,x)\ . \tag{1.7}$$

Explicitly show that this is equivalent to $\mathrm{Tr}\,\widehat{A} = \int \mathrm{d}^4p\ \langle p|\,\widehat{A}\,|p\rangle = \int \mathrm{d}^4p\, A(p,p)$. Show that for an operator invariant under translations (cf (1.4)) formally one has

$$\mathrm{Tr}\,F(\widehat{A}) = \delta^{(4)}(0)\ \int \mathrm{d}^4p\, F(A(p))\ . \tag{1.8}$$

Interpret $\delta^{(4)}(0)$ as $\langle p|\,p\rangle$ and argue that if one replaces $\mathbf{R}^4$ by a hypercube of volume L^4, this factor would be replaced according to $\delta^{(4)}(0)\ \to\ \left(\frac{L}{2\pi}\right)^4$.

1.2 Green's function and generating functional in $0+1$ dimensions

Consider the $0+1$ dimensional quantum "field" theory (this is actually quantum mechanics) with action

$$S_0[\phi] = \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{1}{2}\left[\dot{\phi}^2 - \omega^2\phi^2\right]\ , \tag{1.9}$$

where $\dot{\phi} = \frac{\partial\phi}{\partial t} \equiv \partial_t^2\phi$.

a) Determine the corresponding Green's functions $G(t)$ as the solutions of

$$\left(\partial_t^2 + \omega^2\right) G(t) = \delta(t)\ , \tag{1.10}$$

by first solving this equation for all $t \neq 0$ and then matching solutions to get the correct discontinuity at $t=0$ to reproduce the δ-singularity.

b) Obtain Feynman's Green's function from the Fourier transform with the $i\epsilon$ prescription:

$$G_F(t) = \int_{-\infty}^{\infty} \mathrm{d}p^0 \frac{e^{-ip^0t}}{(p^0)^2 + \omega^2 - i\epsilon}\ , \tag{1.11}$$

and compare with the results from a).

c) Define the generating functional as

$$Z_0[J] = \int \mathcal{D}\phi\ \exp\left[iS[\phi] + i\int \mathrm{d}t\, J(t)\phi(t)\right] \tag{1.12}$$

and compute $Z_0[J]/Z_0[0]$. Explicitly convince yourself that

$$\frac{1}{Z_0[0]}\frac{\delta}{\delta J(t_1)}\cdots\frac{\delta}{\delta J(t_4)} Z_0[J]\Big|_{J=0} \tag{1.13}$$

gives the functional integal representation of the (free) four-point function $\langle T[\phi(t_1)\ldots\phi(t_4)]\rangle_0$.

d) Now add to the action S_0 an interaction term $S = S_0 + S_{\mathrm{int}}$ with $S_{\mathrm{int}} = \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{g}{6}\phi^3$. Define $Z[J]$ as $Z_0[J]$ but with S_0 replaced by S. Compute $Z[J]$ and $Z[0]$ to second order in g. Give the graphical representation of the terms you find.

e) Deduce the full 2-point function as

$$\langle T[\phi(t_1)\phi(t_2)]\rangle = -\frac{1}{Z_0[0]}\frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)}Z_0[J]\Big|_{J=0} . \tag{1.14}$$

Discuss the graphical representation of the terms you get and observe that the vacuum bubbles have cancelled.

1.3 $Z[J]$, $W[J]$ and $\Gamma[\varphi]$ in ϕ^4-theory

Consider a quantum field theory with action

$$S = \int \mathrm{d}^4x \big[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{24}\phi^4\big] . \tag{1.15}$$

Explicitly compute the functionals $Z[J]$, $W[J]$ and $\Gamma[\varphi]$ up to first order in the coupling constant g. Interpret the different terms in $\Gamma[\varphi]$.

1.4 Effective action from integrating out fermions in QED

a) Show that in spinor quantum electrodynamics, the functional integral representation of time-ordered expectation values of operators that do not involve the matter fields can be done in two steps, where the first step involves doing the functional integral over the fermions only. This amounts to computing

$$\begin{aligned} S_{\mathrm{vac,vac}}[A] &= \langle \mathrm{vac,\ out}\,|\mathrm{vac,\ in}\rangle_A \\ &= |\mathcal{N}|^2 \int \prod_l \mathcal{D}\psi_l \mathcal{D}\overline{\psi}_l\ \exp\left\{ -i\int \mathrm{d}^4x\, \overline{\psi}(x)\big(\partial\!\!\!/ + m + ieA\!\!\!/(x)\big)\psi(x)\right\} . \end{aligned} \tag{1.16}$$

b) Compute this integral exactly in terms of an appropriate functional determinant. Of course, this determinant is ill-defined and needs regularization. (One could e.g. replace $\mathrm{Det}\,[\ldots] = \prod_n \lambda_n$ by $\prod_{n=0}^N \lambda_n$ and let $N\to\infty$ in the end, or replace it by $\prod_n \lambda_n e^{-\epsilon n^2}$ and let $\epsilon \to 0$ in the end.) Once regularized, it can be manipulated as an ordinary determinant. Show that

$$\frac{S_{\mathrm{vac,vac}}[A]}{S_{\mathrm{vac,vac}}[0]} = \mathrm{Det}\left[1 + \big(\partial\!\!\!/ + m\big)^{-1}\, ieA\!\!\!/(x)\right] . \tag{1.17}$$

Use $\mathrm{Det}\,(1+M) = \exp(\,\mathrm{Tr}\,\log(1+M))$ and the power series expansion of $\log(1+x)$ to get

$$\frac{S_{\mathrm{vac,vac}}[A]}{S_{\mathrm{vac,vac}}[0]} = \exp\left\{ -\,\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}\left[\big(\partial\!\!\!/ + m\big)^{-1}\,(-ie)A\!\!\!/(x)\right]^n\right\} , \tag{1.18}$$

and give a Feynman diagram interpretation of each term in the sum.

c) One can show that each term in the sum in the exponent is purely imaginary which allows to write

$$S_{\mathrm{vac,vac}}[A] = S_{\mathrm{vac,vac}}[0]\ \exp\left\{ i\int \mathrm{d}^4x\ \mathcal{L}_{\mathrm{eff}}^{1-\mathrm{loop}}[A]\right\} , \tag{1.19}$$

where $\mathcal{L}_{\mathrm{eff}}^{1-\mathrm{loop}}[A]$ is the effective action obtained by integrating out the fermions. Show that by taking functional derivatives with respect to $A_{\mu_i}(x_i)$ of $\int \mathrm{d}^4x\ \mathcal{L}_{\mathrm{eff}}^{1-\mathrm{loop}}[A]$ one generates time-ordered (connected) correlation functions of the electromagnetic current operators $J^{\mu_i}(x_i) = -ie\bar{\psi}(x_i)\gamma^{\mu_i}\psi(x_i)$.

2 Exercices : A few results independent of perturbation theory

2.1 Two-particle intermediate states and branch cut of the two-point function

Just as we have shown that a one-particle intermediate state (mass m) yields a pole in the two-point function $\widehat{G}_{(2)}(q_1, q_2)$ at $-q_1^2 = m^2$, one can see that a two-particle intermediate state leads to a branch cut. More precisely, if the intermediate particles have masses m_1 and m_2, the two-point function has a branch cut for $-q_1^2 \geq (m_1 + m_2)^2$. This corresponds to the kinematical condition of having enough energy to produce on-shell particles of masses m_1 and m_2.

a) Recall that, for a complex variable z, the logarithm $\log z$ has a branch cut along the negative real axis with discontinuity being $\log(-\alpha + i\epsilon) - \log(-\alpha - i\epsilon) = 2\pi i$, where $\alpha > 0$. Recall also that $\lim_{\epsilon\to 0} \frac{\epsilon}{(x-x_0)^2+\epsilon^2} = \pi\, \delta(x - x_0)$. Let $g(x)$ be some smooth function defined for real $x \geq 0$ and be such that $g(x) \to 0$ as $x \to \infty$. Let

$$f(z) = \int_0^\infty \mathrm{d}x\, \frac{g(x)}{x - z} \ . \tag{2.1}$$

Show that this a well-defined function on the complex z-plane, except on the positive real z-axis where f has a branch cut with discontinuity

$$f(\alpha + i\epsilon) - f(\alpha - i\epsilon) = 2\pi i\, g(\alpha) \ , \quad \alpha > 0 \ . \tag{2.2}$$

b) Let now

$$f(E) = \int \mathrm{d}^3p\, \frac{g(\vec{p})}{E - \omega_1(\vec{p}) - \omega_2(\vec{p})} \ , \qquad \omega_a(\vec{p}) = \sqrt{\vec{p}^{\,2} + m_a^2} \ , \tag{2.3}$$

with some smooth function $g(\vec{p})$ such that $\vec{p}^{\,2} g(\vec{p}) \to 0$ as $\vec{p}^{\,2} \to \infty$. Show that this is a well-defined function on the complex E-plane, except for a branch cut for real E when $E \geq m_1 + m_2$, with a discontinuity given by

$$f(E + i\epsilon) - f(E - i\epsilon) = -2\pi i\, \frac{p\, \omega_1(p)\omega_2(p)}{E}\, \widetilde{g}(p)\Big|_{p = p_{\mathrm{cm}}(E, m_1, m_2)} \ , \quad \text{for } E \geq m_1 + m_2 \ , \tag{2.4}$$

where $p_{\mathrm{cm}}(E, m_1, m_2) = \big(E^4 + (m_1^2 - m_2^2)^2 - 2E^2(m_1^2 + m_2^2)\big)/(4E^2)$ is the function determining the center-of-mass momentum and $\widetilde{g}(p) = \int \mathrm{d}\Omega\, g(\vec{p})\big|_{|\vec{p}|=p}$ is the angular integral of g.

c) Repeat the arguments that led to equation (2.9) of the lecture notes, but now for two-particle intermediate states where the two particles have masses m_1 and m_2. To simplify the argument, choose a Lorentz frame where $\vec{q}_1 = 0$ and call $q_1^0 = E$. Use the results from part b) to show that the two-particle intermediate states lead to a branch cut for $E \geq m_1 + m_2$ or, going back to an arbitrary Lorentz frame, for $-q_1^2 \geq (m_1 + m_2)^2$.

2.2 A theory of two scalars, one charged and one neutral

Consider the Lagrangian

$$\mathcal{L} = -\partial_\nu \phi_B \partial^\nu \phi_B^* - M_B^2 \phi_B \phi_B^* - \frac{1}{2}\partial_\nu \varphi_B \partial^\nu \varphi_B - \frac{1}{2} m_B^2 \varphi_B^2 - g_B \phi_B \phi_B^* \varphi_B \ . \tag{2.5}$$

a) Rewrite $\mathcal{L}$ in terms of renomalized fields, masses and coupling (let $g_B = \frac{Z_g}{Z_\phi \sqrt{Z_\varphi}} g$) and identify the counterterms.

b) Determine the complete propagators Δ'_ϕ and Δ'_φ in terms of the 1PI propagators Π^*_ϕ and Π^*_φ.

c) Draw the one-loop contributions to Π^*_ϕ and Π^*_φ and identify the contributions of the counterterms.

d) Discuss how to renormalize the Π^* and Δ' for both types of fields.

3 Exercices : one-loop radiative corrections

3.1 The simplest (non-trivial) one-loop integral

The simplest, non-trivial one-loop integral consists of a loop of a scalar particle with two external lines attached. This would be encountered when computing Π^* in a scalar φ^3 theory or when computing the one-loop corrections to the coupling in scalar ϕ^4 theory. In dimensional regularization it is given by

$$I(P) = i\, \widetilde{\mu}^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^4} \frac{1}{(k^2+m^2-i\epsilon)((P-k)^2+m^2-i\epsilon)} \ , \tag{3.1}$$

where we did not include the coupling constants or the symmetry factor $\frac{1}{2}$ into the definition of $I(P)$, but we did include a factor of i for convenience.

a) Argue that the factor $\widetilde{\mu}^{4-d}$ keeps $I(P)$ dimensionless. Show that

$$I(P) = -\frac{\pi^{d/2}}{(2\pi)^4}\Gamma(2-\frac{d}{2}) \int_0^1 \mathrm{d}x \left[\frac{x(1-x)P^2+m^2}{\widetilde{\mu}^2}\right]^{\frac{d}{2}-2} \ . \tag{3.2}$$

b) Expanding in $\epsilon = 4-d$, show that

$$I(P) = -\frac{1}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma - \log\pi - \int_0^1 \mathrm{d}x \log\left[\frac{x(1-x)P^2+m^2}{\widetilde{\mu}^2}\right] + \mathcal{O}(\epsilon)\right) \ . \tag{3.3}$$

3.2 Vacuum polarization in QED

The one-loop vacuum-polarization in QED is given by

$$\Pi^{*\mu\nu}_{\text{loop},e^2}(q) = \frac{-i\,(e\widetilde{\mu}^{4-d})^2}{(2\pi)^4}\int \mathrm{d}^4k \frac{\mathrm{tr}\,\left[(-i\not{k}+m)\gamma^\mu(-i(\not{k}-\not{q})+m)\gamma^\nu\right]}{[k^2+m^2-i\epsilon][(k-q)^2+m^2-i\epsilon]}\ . \tag{3.4}$$

a) Show that after evaluating the Dirac trace in the numerator (as in 4 dimensions), doing the Feynman trick and shifting the integration variables $k-xq\to k$ one gets

$$\Pi^{*\mu\nu}_{\text{loop},e^2}(q) = \frac{-4i\,e^2}{(2\pi)^4}(\widetilde{\mu}^2)^{4-d}\int_0^1 \mathrm{d}x\int \mathrm{d}^dk\Big[-(k+xq)^\mu(k-(1-x)q)^\nu-(k+xq)^\nu(k-(1-x)q)^\mu$$
$$+(k+xq)(k-(1-x)q)\eta^{\mu\nu}+m^2\eta^{\mu\nu}\Big]\frac{1}{\left[k^2+m^2+x(1-x)q^2\right]^2}\ . \tag{3.5}$$

b) Do the Wick rotation, also continuing the external q^μ to a Euclidean q_E^μ. Recall that, due to rotational symmetry, terms linear in k^μ don't contribute to the integral, while terms $k^\mu\,k^\nu$ can be replaced by $\frac{1}{d}\,\eta^{\mu\nu}k^2$ which become $\frac{1}{d}\,\eta^{\mu\nu}k_E^2$ after the continuation. Show that this yields

$$\Pi^{*\mu\nu}_{\text{loop},e^2}(q_E) = \frac{-4i\,e^2}{(2\pi)^4}(\widetilde{\mu}^2)^{4-d}\int_0^1 \mathrm{d}x\ i\int \mathrm{d}^dk_E\frac{\left[m^2-x(1-x)q_E^2+(1-\frac{2}{d})k_E^2\right]\eta^{\mu\nu}+2x(1-x)q_E^\mu q_E^\nu}{\left[k_E^2+m^2+x(1-x)q_E^2\right]^2}\ . \tag{3.6}$$

which evaluates to ($R^2=m^2+x(1-x)q_E^2$)

$$\Pi^{*\mu\nu}_{\text{loop},e^2}(q_E) = \frac{4e^2}{(2\pi)^4}\int_0^1 \mathrm{d}x\Bigg\{\Big[\big[m^2-x(1-x)q_E^2\big]\eta^{\mu\nu}+2x(1-x)q_E^\mu q_E^\nu\Big]\pi^{d/2}\,\Gamma\Big(2-\frac{d}{2}\Big)$$
$$+\Big(1-\frac{2}{d}\Big)\,\eta^{\mu\nu}\,\pi^{d/2}\,\frac{\Gamma(\frac{d}{2}+1)\Gamma(1-\frac{d}{2})}{\Gamma(\frac{d}{2})}\,R^2\Bigg\}\left(\frac{R^2}{\widetilde{\mu}^2}\right)^{\frac{d}{2}-2}\ . \tag{3.7}$$

c) Observing that $(1-\frac{2}{d})\frac{\Gamma(\frac{d}{2}+1)\Gamma(1-\frac{d}{2})}{\Gamma(\frac{d}{2})} = (1-\frac{2}{d})\,\frac{d}{2}\,\Gamma(1-\frac{d}{2}) = (\frac{d}{2}-1)\Gamma(1-\frac{d}{2}) = -\Gamma(2-\frac{d}{2})$ show that finally (upon changing back the argument from q_E to q)

$$\Pi^{*\mu\nu}(q) = \left(q^2\eta^{\mu\nu}-q^\mu q^\nu\right)\pi(q^2)\ , \tag{3.8}$$

with

$$\pi_{\text{loop},e^2}(q^2) = -\frac{8e^2}{(2\pi)^4}\Gamma\Big(2-\frac{d}{2}\Big)\,\pi^{d/2}\int_0^1 \mathrm{d}x\,x(1-x)\left[\frac{m^2+x(1-x)q^2}{\widetilde{\mu}^2}\right]^{\frac{d}{2}-2}\ . \tag{3.9}$$

d) Set $d=4-\epsilon$ and expand the result in ϵ. Recall that $\Gamma(2-\frac{d}{2})=\Gamma(\frac{\epsilon}{2})=\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)$, and also $a^\epsilon=e^{\epsilon\log a}=1+\epsilon\log a+\mathcal{O}(\epsilon^2)$. Show that

$$\pi_{\text{loop},e^2}(q^2) = -\frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon}-\frac{\gamma}{2}-\frac{1}{2}\log\pi-3\int_0^1 \mathrm{d}x\,x(1-x)\log\left[\frac{m^2+x(1-x)q^2}{\widetilde{\mu}^2}\right]+\mathcal{O}(\epsilon)\right)\ . \tag{3.10}$$

4 Exercices : general renormalization theory

4.1 Yukawa theory

Consider a 4-dimensional quantum field theory of a massive Dirac fermion ψ coupled to a massive scalar field ϕ via a Yukawa coupling $g\bar{\psi}\psi\phi$.

a) Determine whether this theory is renormalizable. Let $\Gamma^{n,m}$ be the one-particle irreducible $n+m$ point function with n external fermions and m external scalars. Determine which $\Gamma^{n,m}$ are superficially divergent and which ones are superficially convergent. Draw the one-, two-, and three-loop contributions to $\Gamma^{0,2}$ and $\Gamma^{2,1}$ and explicitly (by looking at the propagators and integrations) determine their superficial degree of divergence.

b) What happens if one replaces the Yukawa coupling by a derivative coupling $g'\bar{\psi}\gamma^\mu\psi\partial_\mu\phi$? Study the superficial degree of divergences of $\Gamma^{4,0}$ at one, two and three loops.

4.2 Quantum gravity

The Einstein-Hilbert action for gravity in $d \geq 3$ dimensions is

$$S_{\text{EH}} = \frac{1}{\kappa^2}\int \mathrm{d}^d x\,\sqrt{g}\,R[g] \tag{4.1}$$

where $R[g]$ is the scalar (Ricci) curvature computed from the metric $g_{\mu\nu}$ and $\sqrt{g}$ denotes the square-root of the determinant of the metric. In perturbative quantum gravity one lets $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with “small” $h_{\mu\nu}$ and does a perturbative expansion of the action. Show that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)$ where one raises and lowers indices on h with η. Also show that $\sqrt{g} = 1 + \frac{1}{2}h^\mu_\mu$. (Hint: use the formula $\det g = \exp(\,\mathrm{tr}\,\log g)$. Recall the formula for the Christoffel symbols and show that

$$2\Gamma^\rho_{\mu\nu} = \partial_\mu h^\rho_\nu + \partial_\nu h^\rho_\mu - \partial^\rho h_{\mu\nu} - h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \mathcal{O}(h^h\partial h)\ . \tag{4.2}$$

Recall that

$$R^\rho{}_{\nu\lambda\mu} = \partial_\lambda\Gamma^\rho_{\mu\nu} - \partial\Gamma^\rho_{\lambda\nu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} \quad , \quad R_{\nu\mu} = R^\rho{}_{\nu\rho\mu} \quad , \quad R = g^{\mu\nu}R_{\nu\mu}\ , \tag{4.3}$$

and argue *without keeping track of the indices* that $\partial\Gamma \sim \partial^2 h + h\partial^2 h + (\partial h)^2 + h^2\partial^2 h + h(\partial h)^2 + \ldots$. Give the corresponding expansions (still without keeping track of the indices) of $R^\rho{}_{\nu\lambda\mu}$, R and finally of $\sqrt{g}R$. Deduce the large momentum behaviour of the h-propagator as well as the dimensions of the various interactions in space-time dimension d. Discuss whether this theory is renormalizable in d dimensions? What happens in $d = 2$ dimensions ?

4.3 Non-renormalizable interactions from integrating out a massive field

Consider a theory of two scalars φ and ϕ with masses m and M where $M \gg m$. Let their action be

$$S[\varphi,\phi] = \int \mathrm{d}^4 x\big(-\frac{1}{2}\big)\big[(\partial\varphi)^2 + m^2\varphi^2 + (\partial\phi)^2 + M^2\phi^2 + gm\varphi\phi^2\big]\ . \tag{4.4}$$

(We have defined the coupling as $g \times m$ to have a dimensionless g.) Define the effective action for the light field φ as

$$e^{iS_{\rm eff}[\varphi]} = \int \mathcal{D}\phi \, e^{iS[\varphi,\phi]} \ . \tag{4.5}$$

Show that the resulting determiant can be interpreted as a sum of one-loop diagrams with N ϕ-propagators and N external φ attached at the vertices. Compute the determinant by neglecting the ϕ momenta with respect to M and show that this results in infinitely many new couplings for φ of the form

$$\sum_{N=1}^{\infty} \frac{(-)^N}{N} \Lambda^4 \, \left(\frac{m\, g}{M^2}\right)^N \varphi^N \ , \tag{4.6}$$

where Λ is some large UV-cutoff which one can take of the order of M. Discuss the role of these new interactions for $N \leq 4$ and for $N > 4$.

5 Exercices : Renormalization group and Callan-Szymanzik equations

5.1 β-function in ϕ^3-theory in 6 dimensions

Consider a scalar field ϕ of mass m in $d = 6$ with a cubic self-interaction $\frac{g}{6}\phi^3$.

a) Write the action in terms of the bare quantities and then in terms of the renormalized ones, thus determining explicitly the form of the counter-terms. In particular one lets $g_B Z^{3/2} = g Z_g$. Give the Feynman rules for the counter-term vertices. Discuss how Z_g is fixed by some appropriate renomalization condition imposed on the proper 3-point vertex $\Gamma^{(3)}(p_1, p_2, -p_1, -p_2)$.

b) Compute this proper 3-point vertex $\Gamma^{(3)}(p_1, p_2, -p_1, -p_2)$ up to order g^3. Impose that this $\Gamma^{(3)}$ equals g for some particular momentum configuration at scale μ. (One could conveneiently choose $p_1^2 = p_2^2 = \mu^2$ and $p_1 \cdot p_2 = -\mu^2/2$ so that $p_3^2 = (p_1 + p_2)^2 = \mu^2$, too.)

c) Show how to compute $g(\widetilde{\mu})$ at some slightly different scale. Obtain the one-loop β-function (in a region where $\mu \gg m$), and determine the running of the coupling given by $\frac{{\rm d}g(\mu)}{{\rm d}\log\mu} = \beta(g(\mu))$.

5.2 3-point vertex function in an asymptotically free theory

Non-abelian gauge theories have a cubic self-coupling of the gauge field with coupling constant g. Assume that the β-function is given by $\beta(g) = -bg^2$ with $b > 0$ (at one loop), and the η-function for the gauge field by $\eta = ag^2$.

a) Solve the running of the coupling constant and give the explicit form of $g(\mu)$ for large μ.

b) Write and solve the Callan-Szymanzik equation for the proper vertex function of 3 gauge fields $\Gamma^{\mu,\nu\rho}(p_j, g, \mu)$. Use the solution to determine the asymptotics of $\Gamma^{\mu,\nu\rho}$ in the large momentum limit $p_j = \lambda q_j$ with q_j fixed and $\lambda \to \infty$.

6 Exercices : Non-abelian gauge theories

6.1 Some one-loop vertex functions

Use the action and Feynman rules as given in the lecture.

a) Draw all Feynman diagrams contributing to the 1PI gauge-boson-ghost-antighost vertex function $\Gamma^{\mu}_{\alpha\beta\gamma}(p_1, p_2, p_3)$ at one loop (2 diagrams). For each diagram determine the superficial degree of divergence.

b) Same question for the 1PI three gauge-boson vertex function $\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(p_1, p_2, p_3)$ at one loop. (There are 8 diagrams if one counts as different diagrams two fermion loops with opposite orientations of the arrows.)

c) Same question for the 1PI vertex function for two ghosts and two antighosts $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$. (There are 4 diagrams.)

6.2 More general gauge-fixing functions

Instead of the gauge-fixing function $f^{\alpha} = \partial_{\mu} A^{\alpha\mu}$ now choose $f^{\alpha} = (D_{\mu} A^{\mu})^{\alpha}$.

a) Give the explicit form of the corresponding $\mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh}$ and deduce the corresponding vertices involving the ghost and antighost fields. In particular give the vertex between the ghost, antighost and two gauge bosons.

b) Show that, in addition to the diagrams present in c) of the previous exercice, there are new types of Feynman diagrams contributing to the one-loop 4-point 2 ghost- 2 antighost vertex function $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$. Determine their superficial degrees of divergence. Does it come as a surprise if you find the same answer as in c) of the previous exercice?

6.3 Two ghost - two antighost counterterms

In the Lagrangian used in exercice 6.1 there are no terms $\sim \omega^* \omega^* \omega \omega$, and thus we expect that one should not need such counterterms either. This would mean that the vertex function $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ given in c) of this exercice should not be divergent, although it had a superficial degree of divergence equal to 0. Write the expression for the 4 corresponding Feynman diagrams and try to argue that there are cancellations between the diagrams.