Dunkl Operator and Quantization of Orbifolds

Xiang Tang

Washington University at St. Louis

February 17th 2020, International Solvay Institutes

Goal :

In this talk, we will explain our some of our recent works about understanding quantization of orbifolds and its relation to deformation of singularities. Dunkl operator leads us to some very interesting construction.

Plan of this talk

1. Orbifold and deformation quantization
2. Hochschild cohomology of an orbifold algebra
3. Dunkl operator and a construction for $\mathbb{Z}_2$ orbifolds

In this part, we will briefly introduce a noncommutative geometry approach to study an orbifold. We will explain the problem of deformation quantization of an orbifold algebra.

Orbifold

An orbifold is a separable Hausdorff topological space which is locally modeled on the quotient of $\mathbb{R}^n$ by a linear action of a finite group. Such a topological space is very important in both mathematics and physics.

Example

Orbifold

An orbifold is a separable Hausdorff topological space which is locally modeled on the quotient of $\mathbb{R}^n$ by a linear action of a finite group. Such a topological space is very important in both mathematics and physics.

Example

1. $\mathbb{C}^2/\mathbb{Z}_2$, and $\mathbb{C}^n/\mathbb{Z}_n$.

Orbifold

An orbifold is a separable Hausdorff topological space which is locally modeled on the quotient of $\mathbb{R}^n$ by a linear action of a finite group. Such a topological space is very important in both mathematics and physics.

Example

1. $\mathbb{C}^2/\mathbb{Z}_2$, and $\mathbb{C}^n/\mathbb{Z}_n$.
2. tear drop

Orbifold

An orbifold is a separable Hausdorff topological space which is locally modeled on the quotient of $\mathbb{R}^n$ by a linear action of a finite group. Such a topological space is very important in both mathematics and physics.

Example

1. $\mathbb{C}^2/\mathbb{Z}_2$, and $\mathbb{C}^n/\mathbb{Z}_n$.
2. tear drop
3. moduli spaces of curves

Noncommutative algebra

Let's look at the example that a finite group Γ acts on a manifold M. When the action is not free, the quotient space M/Γ is an orbifold.

Noncommutative algebra

Let's look at the example that a finite group Γ acts on a manifold M. When the action is not free, the quotient space M/Γ is an orbifold.
Let $\mathcal{O}_M$ be the algebra of functions on M. Γ acts $\mathcal{O}_M$ by translation. We consider an algebra generated by $\mathcal{O}_M$ and $\gamma \in \Gamma$ with the relation

$$\gamma f = \gamma(f)\gamma, \qquad \text{for } f \in \mathcal{O}_M.$$

Noncommutative algebra

Let's look at the example that a finite group Γ acts on a manifold M. When the action is not free, the quotient space M/Γ is an orbifold.
Let $\mathcal{O}_M$ be the algebra of functions on M. Γ acts $\mathcal{O}_M$ by translation. We consider an algebra generated by $\mathcal{O}_M$ and $\gamma \in \Gamma$ with the relation

$$\gamma f = \gamma(f)\gamma, \qquad \text{for } f \in \mathcal{O}_M.$$

This algebra is denoted by $\mathcal{O} \rtimes \Gamma$, which is a noncommutative algebra associated to the Γ action on M.

Noncommutative algebra

Let's look at the example that a finite group Γ acts on a manifold M. When the action is not free, the quotient space M/Γ is an orbifold.
Let $\mathcal{O}_M$ be the algebra of functions on M. Γ acts $\mathcal{O}_M$ by translation. We consider an algebra generated by $\mathcal{O}_M$ and $\gamma \in \Gamma$ with the relation

$$\gamma f = \gamma(f)\gamma, \qquad \text{for } f \in \mathcal{O}_M.$$

This algebra is denoted by $\mathcal{O} \rtimes \Gamma$, which is a noncommutative algebra associated to the Γ action on M.

Example

When M is a vector space V, then the noncommutative algebra is $S(V^*) \rtimes \Gamma$.

In general, given an orbifold X, one may find different representations of it by different group actions. The associated noncommutative algebras are all Morita equivalent. For our talk today, we will focus on the example $\mathcal{O}_M \rtimes \Gamma$. And most of our results generalize to general orbifold algebras.

Orbifold algebra

In general, given an orbifold X, one may find different representations of it by different group actions. The associated noncommutative algebras are all Morita equivalent. For our talk today, we will focus on the example $\mathcal{O}_M \rtimes \Gamma$. And most of our results generalize to general orbifold algebras.

The orbifold algebra $\mathcal{O}_M \rtimes \Gamma$ contains numerous information of the orbifold M/Γ. For example, the K-theory and cohomology of M/Γ are isomorphic to the K-theory and cohomology of $\mathcal{O}_M \rtimes \Gamma$.

Symplectic Manifold and quantization

A symplectic manifold M is the phase space of a classical mechanic system. The physical observables of this system are functions on the symplectic manifold.

Symplectic Manifold and quantization

A symplectic manifold M is the phase space of a classical mechanic system. The physical observables of this system are functions on the symplectic manifold.
A quantum mechanic system is described by a Hilbert space H. The physical observables are represented by self-adjoint operators on the Hilbert space.

Symplectic Manifold and quantization

A symplectic manifold M is the phase space of a classical mechanic system. The physical observables of this system are functions on the symplectic manifold.
A quantum mechanic system is described by a Hilbert space H. The physical observables are represented by self-adjoint operators on the Hilbert space.
A "quantization map" relating a classical mechanic system to its quantum version can be described by a linear map

$$Q : \mathcal{O}_M \to Op(H).$$

The "pull back" of the operator product to $\mathcal{O}_M$ defines a new associative product.

Formal deformation quantization and Moyal-Weyl product

The above idea of quantization can be formulated in the framework of deformation quantization introduced by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in 1977.

Formal deformation quantization and Moyal-Weyl product

The above idea of quantization can be formulated in the framework of deformation quantization introduced by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in 1977.
A *formal deformation quantization* of a Poisson manifold (M, ω) is an associative product $*_\hbar$ on $C^\infty(M)[[\hbar]]$, such that
(i) $f *_\hbar g = fg + \hbar\{f, g\} + \sum_{i\geq 2} \hbar^i C_i(f, g)$,
(ii) C_i*'s are bilinear local differential operators.*

Formal deformation quantization and Moyal-Weyl product

The above idea of quantization can be formulated in the framework of deformation quantization introduced by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in 1977.
A *formal deformation quantization* of a Poisson manifold (M, ω) is an associative product $*_\hbar$ on $C^\infty(M)[[\hbar]]$, such that
(i) $f *_\hbar g = fg + \hbar\{f, g\} + \sum_{i\geq 2} \hbar^i C_i(f, g)$,
(ii) C_i*'s are bilinear local differential operators.*
De Wilde and Lecome solved the existence of a formal deformation quantization on a symplectic manifold in 1983.

Formal deformation quantization and Moyal-Weyl product

The above idea of quantization can be formulated in the framework of deformation quantization introduced by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in 1977.
A *formal deformation quantization* of a Poisson manifold (M, ω) is an associative product $*_\hbar$ on $C^\infty(M)[[\hbar]]$, such that
(i) $f *_\hbar g = fg + \hbar\{f, g\} + \sum_{i\geq 2} \hbar^i C_i(f, g)$,
(ii) C_i*'s are bilinear local differential operators.*
De Wilde and Lecome solved the existence of a formal deformation quantization on a symplectic manifold in 1983.
Kontsevich solved both the existence and classification problems of formal deformation quantizations on a Poisson manifold in 1997.

Examples of deformation quantization

In the case of $\mathbb{R}^2$ with the standard symplectic 2-form $dx^1 \wedge dx^2$, the standard quantization procedure defines a product on $C^\infty(\mathbb{R}^2)[[\hbar]]$,

Examples of deformation quantization

In the case of $\mathbb{R}^2$ with the standard symplectic 2-form $dx^1 \wedge dx^2$, the standard quantization procedure defines a product on $C^\infty(\mathbb{R}^2)[[\hbar]]$,

$$f \star g(x) = \exp(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j})f(y,\hbar)g(z,\hbar)|_{x=y=z},$$

where ω^{ij} is the inverse of the symplectic matrix ω.

Examples of deformation quantization

In the case of $\mathbb{R}^2$ with the standard symplectic 2-form $dx^1 \wedge dx^2$, the standard quantization procedure defines a product on $C^\infty(\mathbb{R}^2)[[\hbar]]$,

$$f \star g(x) = \exp(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j})f(y,\hbar)g(z,\hbar)|_{x=y=z},$$

where ω^{ij} is the inverse of the symplectic matrix ω.

In the case of dual of a Lie algebra $\mathfrak{g}$, the universal enveloping algebra can be viewed as a formal deformation of $S(\mathfrak{g})$.

A *formal deformation quantization* of $(A, [\Pi])$ is an associative product

Deformation quantization of orbifold algebra

A *formal deformation quantization* of $(A, [\Pi])$ is an associative product

$$\star : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]], \quad (a_1, a_2) \mapsto a_1 \star a_2 = \sum_{k=0}^{\infty} \hbar^k c_k(a_1, a_2)$$

satisfying the following properties :

Deformation quantization of orbifold algebra

A *formal deformation quantization* of $(A, [\Pi])$ is an associative product

$$\star : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]], \quad (a_1, a_2) \mapsto a_1 \star a_2 = \sum_{k=0}^{\infty} \hbar^k c_k(a_1, a_2)$$

satisfying the following properties :

1. Each one of the maps $c_k : A[[\hbar]] \otimes A[[\hbar]] \to A[[\hbar]]$ is $\mathbb{C}[[\hbar]]$-bilinear ;

Deformation quantization of orbifold algebra

A *formal deformation quantization* of $(A, [\Pi])$ is an associative product

$$\star : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]], \quad (a_1, a_2) \mapsto a_1 \star a_2 = \sum_{k=0}^{\infty} \hbar^k c_k(a_1, a_2)$$

satisfying the following properties :

1. Each one of the maps $c_k : A[[\hbar]] \otimes A[[\hbar]] \to A[[\hbar]]$ is $\mathbb{C}[[\hbar]]$-bilinear ;
2. One has $c_0(a_1, a_2) = a_1 \cdot a_2$ for all $a_1, a_2 \in A$;

A *formal deformation quantization* of $(A, [\Pi])$ is an associative product

$$\star : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]], \quad (a_1, a_2) \mapsto a_1 \star a_2 = \sum_{k=0}^{\infty} \hbar^k c_k(a_1, a_2)$$

satisfying the following properties :

1. Each one of the maps $c_k : A[[\hbar]] \otimes A[[\hbar]] \to A[[\hbar]]$ is $\mathbb{C}[[\hbar]]$-bilinear ;
2. One has $c_0(a_1, a_2) = a_1 \cdot a_2$ for all $a_1, a_2 \in A$;
3. The relation

$$a_1 \star a_2 - c_0(a_1, a_2) - \frac{i}{2}\hbar\Pi(a_1, a_2) \in \hbar^2 A[[\hbar]].$$

Part II : Hochschild cohomology of orbifold algebras

In this part, we will discuss some Hochschild cohomology results of orbifold algebras and explain its connections to formal deformation quantization.

An example of twisted derivation

Consider the $\mathbb{Z}_2 = \{1, e\}$ action on $\mathbb{R}$ by reflection, i.e. $e : x \mapsto -x$. Then e lifts to act on $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by

$$\hat{e}(f)(x) = f(-x).$$

An example of twisted derivation

Consider the $\mathbb{Z}_2 = \{1, e\}$ action on $\mathbb{R}$ by reflection, i.e. $e : x \mapsto -x$. Then e lifts to act on $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by

$$\hat{e}(f)(x) = f(-x).$$

We are interested in linear operators on $C^\infty(\mathbb{R})$ satisfying the generalized Leibniz rule

$$\tilde{D}(fg) = f\tilde{D}(g) + \tilde{D}(f)\hat{e}(g).$$

An example of twisted derivation

Consider the $\mathbb{Z}_2 = \{1, e\}$ action on $\mathbb{R}$ by reflection, i.e. $e : x \mapsto -x$. Then e lifts to act on $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by

$$\hat{e}(f)(x) = f(-x).$$

We are interested in linear operators on $C^\infty(\mathbb{R})$ satisfying the generalized Leibniz rule

$$\tilde{D}(fg) = f\tilde{D}(g) + \tilde{D}(f)\hat{e}(g).$$

Example

$\tilde{D}(f)(x) = \frac{f(x)-f(-x)}{x}$.

Note : $\tilde{D}^2(f) = 0$.

Hochschild cohomology

Let A be an algebra over a field k, and M be a A-bimodule.

Hochschild cohomology

Let A be an algebra over a field k, and M be a A-bimodule.

Definition

Define $C^k(A;M)$ to be $Hom(A^{\otimes k};M)$, and a differential $\partial : C^k(A;M) \longrightarrow C^{k+1}(A;M)$ by

$$\partial(\varphi)(a_1,\cdots,a_{k+1}) = a_1\varphi(a_2,\cdots,a_{k+1}) - \varphi(a_1a_2,\cdots,a_{k+1}) + \cdots + (-1)^i\varphi(a_1,\cdots,a_ia_{i+1},\cdots,a_{k+1}) + \cdots + \varphi(a_1,\cdots,a_k)a_{k+1}.$$

Hochschild cohomology

Let A be an algebra over a field k, and M be a A-bimodule.

Definition

Define $C^k(A;M)$ to be $Hom(A^{\otimes k};M)$, and a differential $\partial : C^k(A;M) \longrightarrow C^{k+1}(A;M)$ by

$$\partial(\varphi)(a_1,\cdots,a_{k+1}) = a_1\varphi(a_2,\cdots,a_{k+1}) - \varphi(a_1a_2,\cdots,a_{k+1})+ \cdots + (-1)^i\varphi(a_1,\cdots,a_ia_{i+1},\cdots,a_{k+1}) + \cdots + \varphi(a_1,\cdots,a_k)a_{k+1}.$$

The Hochschild cohomology $H^\bullet(A;M)$ is defined to be the cohomology of $(C^\bullet(A;M),\partial)$.

Example

Hochschild cohomology

Let A be an algebra over a field k, and M be a A-bimodule.

Definition

Define $C^k(A;M)$ to be $Hom(A^{\otimes k};M)$, and a differential $\partial : C^k(A;M) \longrightarrow C^{k+1}(A;M)$ by

$$\partial(\varphi)(a_1, \cdots, a_{k+1}) = a_1\varphi(a_2, \cdots, a_{k+1}) - \varphi(a_1a_2, \cdots, a_{k+1}) + \cdots + (-1)^i\varphi(a_1, \cdots, a_ia_{i+1}, \cdots, a_{k+1}) + \cdots + \varphi(a_1, \cdots, a_k)a_{k+1}.$$

The Hochschild cohomology $H^\bullet(A;M)$ is defined to be the cohomology of $(C^\bullet(A;M), \partial)$.

Example

1. $H^0(A;M) = \{m \in M |\ am = ma\}$.

Hochschild cohomology

Let A be an algebra over a field k, and M be a A-bimodule.

Definition

Define $C^k(A; M)$ to be $Hom(A^{\otimes k}; M)$, and a differential $\partial : C^k(A; M) \longrightarrow C^{k+1}(A; M)$ by

$$\partial(\varphi)(a_1, \cdots, a_{k+1}) = a_1\varphi(a_2, \cdots, a_{k+1}) - \varphi(a_1a_2, \cdots, a_{k+1}) + \cdots + (-1)^i\varphi(a_1, \cdots, a_ia_{i+1}, \cdots, a_{k+1}) + \cdots + \varphi(a_1, \cdots, a_k)a_{k+1}.$$

The Hochschild cohomology $H^\bullet(A; M)$ is defined to be the cohomology of $(C^\bullet(A; M), \partial)$.

Example

1. $H^0(A; M) = \{m \in M |\ am = ma\}$.
2. $H^1(A; M) = \{\varphi \in Hom_k(A; M) | a\varphi(b) + \varphi(a)b = \varphi(ab)\}$.

Hochschild cohomology of a smooth manifold

Let M be a smooth manifold, and $\mathcal{O}_M$ be the algebra of smooth functions on M.

Hochschild cohomology of a smooth manifold

Let M be a smooth manifold, and $\mathcal{O}_M$ be the algebra of smooth functions on M.

Theorem (Hochschild-Kostant-Rosenberg)

$$H^\bullet(\mathcal{O}_M; \mathcal{O}_M) \cong \Gamma^\infty(\wedge^\bullet TM).$$

Hochschild cohomology of a smooth manifold

Let M be a smooth manifold, and $\mathcal{O}_M$ be the algebra of smooth functions on M.

Theorem (Hochschild-Kostant-Rosenberg)

$$H^\bullet(\mathcal{O}_M; \mathcal{O}_M) \cong \Gamma^\infty(\wedge^\bullet TM).$$

Let Γ be a finite group acting on M. For $\gamma \in \Gamma$, define an $\mathcal{O}_M$-bimodule $\mathcal{O}_M^\gamma$ by

$$(a \cdot \xi \cdot b)(x) := a(x)\xi(x)\gamma(b)(x).$$

Hochschild cohomology of a smooth manifold

Let M be a smooth manifold, and $\mathcal{O}_M$ be the algebra of smooth functions on M.

Theorem (Hochschild-Kostant-Rosenberg)

$$H^\bullet(\mathcal{O}_M;\mathcal{O}_M) \cong \Gamma^\infty(\wedge^\bullet TM).$$

Let Γ be a finite group acting on M. For $\gamma \in \Gamma$, define an $\mathcal{O}_M$-bimodule $\mathcal{O}_M^\gamma$ by

$$(a \cdot \xi \cdot b)(x) := a(x)\xi(x)\gamma(b)(x).$$

Theorem (Neumaier-Pflaum-Posthuma-T)

$$H^\bullet(\mathcal{O}_M;\mathcal{O}_M^\gamma) \cong \Gamma(\wedge^{\bullet-\ell}TM^\gamma \otimes \wedge^\ell N^\gamma).$$

Hochschild cohomology of orbifolds

Let M be a smooth manifold and Γ be a finite group acting on M by diffeomorphisms.

Hochschild cohomology of orbifolds

Let M be a smooth manifold and Γ be a finite group acting on M by diffeomorphisms.

Definition

We consider the orbifold $X = M/\Gamma$. Let M^γ be the γ fixed point submanifold of M. And Γ acts on $\coprod_{\gamma\in\Gamma} M^\gamma$ by $\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha(x))$. The associated inertia orbifold is the quotient space $IX = \coprod_{\gamma\in\Gamma} M^\gamma/\Gamma$.

Hochschild cohomology of orbifolds

Let M be a smooth manifold and Γ be a finite group acting on M by diffeomorphisms.

Definition

We consider the orbifold $X = M/\Gamma$. Let M^γ be the γ fixed point submanifold of M. And Γ acts on $\coprod_{\gamma\in\Gamma} M^\gamma$ by $\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha(x))$. The associated inertia orbifold is the quotient space $IX = \coprod_{\gamma\in\Gamma} M^\gamma/\Gamma$.

Theorem (Neumaier-Pflaum-Posthuma-T)

$$H^\bullet(C^\infty(M)\rtimes\Gamma, C^\infty(M)\rtimes\Gamma) = \Big(\bigoplus_{\gamma\in\Gamma}\Gamma(\wedge^{\bullet-\ell}TM^\gamma\otimes\wedge^\ell N^\gamma)\Big)^\Gamma,$$

Hochschild cohomology of orbifolds

Let M be a smooth manifold and Γ be a finite group acting on M by diffeomorphisms.

Definition

We consider the orbifold $X = M/\Gamma$. Let M^γ be the γ fixed point submanifold of M. And Γ acts on $\bigsqcup_{\gamma\in\Gamma} M^\gamma$ by $\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha(x))$. The associated inertia orbifold is the quotient space $IX = \bigsqcup_{\gamma\in\Gamma} M^\gamma/\Gamma$.

Theorem (Neumaier-Pflaum-Posthuma-T)

$$H^\bullet(C^\infty(M) \rtimes \Gamma, C^\infty(M) \rtimes \Gamma) = \Big(\bigoplus_{\gamma\in\Gamma} \Gamma(\wedge^{\bullet-\ell} TM^\gamma \otimes \wedge^\ell N^\gamma)\Big)^\Gamma,$$

where N^γ is the normal bundle of M^γ in M, and ℓ is the codimension of M^γ in M.

Example of $\mathbb{R}/\mathbb{Z}_2$

When $\mathbb{Z}_2$ acts on $\mathbb{R}$ by reflection, $H^k(C^\infty(\mathbb{R}) \rtimes \mathbb{Z}_2, C^\infty(\mathbb{R}) \rtimes \mathbb{Z}_2)$ is computed as follows.

Example of $\mathbb{R}/\mathbb{Z}_2$

When $\mathbb{Z}_2$ acts on $\mathbb{R}$ by reflection, $H^k(C^\infty(\mathbb{R}) \rtimes \mathbb{Z}_2, C^\infty(\mathbb{R}) \rtimes \mathbb{Z}_2)$ is computed as follows.

- $k = 0$, $H^0 = C^\infty(\mathbb{R})^{\mathbb{Z}_2}$;

Example of $\mathbb{R}/\mathbb{Z}_2$

When $\mathbb{Z}_2$ acts on $\mathbb{R}$ by reflection, $H^k(C^\infty(\mathbb{R}) \rtimes \mathbb{Z}_2, C^\infty(\mathbb{R}) \rtimes \mathbb{Z}_2)$ is computed as follows.

- $k = 0$, $H^0 = C^\infty(\mathbb{R})^{\mathbb{Z}_2}$;
- $k = 1$, $H^1 = (C^\infty(\mathbb{R})\frac{d}{dx})^{\mathbb{Z}_2} \oplus \mathbb{C}\tilde{D}$;

Example of $\mathbb{R}/\mathbb{Z}_2$

When $\mathbb{Z}_2$ acts on $\mathbb{R}$ by reflection, $H^k(C^\infty(\mathbb{R}) \rtimes \mathbb{Z}_2, C^\infty(\mathbb{R}) \rtimes \mathbb{Z}_2)$ is computed as follows.

- $k = 0$, $H^0 = C^\infty(\mathbb{R})^{\mathbb{Z}_2}$;
- $k = 1$, $H^1 = (C^\infty(\mathbb{R})\frac{d}{dx})^{\mathbb{Z}_2} \oplus \mathbb{C}\tilde{D}$;
- $k \geq 2$, $H^\bullet = 0$.

Infinitesimal deformation and noncommutative Poisson structure

Let A be an algebra, and $\star$ be a deformation quantization of A. We write

$$a \star b = ab + \hbar m_1(a,b) + \hbar^2 m_2(a,b) + \cdots .$$

Infinitesimal deformation and noncommutative Poisson structure

Let A be an algebra, and $\star$ be a deformation quantization of A. We write

$$a \star b = ab + \hbar m_1(a, b) + \hbar^2 m_2(a, b) + \cdots .$$

The associativity property of $\star$ implies the following identities

$$\begin{aligned} \partial m_1(a, b, c) &= 0 \\ \partial m_2(a, b, c) &= m_1(m_1(a, b), c) - m_1(a, m_1(b, c)) \\ &= [m, m]_G(a, b, c) \end{aligned}$$

Infinitesimal deformation and noncommutative Poisson structure

Let A be an algebra, and $\star$ be a deformation quantization of A. We write

$$a \star b = ab + \hbar m_1(a,b) + \hbar^2 m_2(a,b) + \cdots .$$

The associativity property of $\star$ implies the following identities

$$\begin{aligned}\partial m_1(a,b,c) &= 0\\ \partial m_2(a,b,c) &= m_1(m_1(a,b),c) - m_1(a, m_1(b,c))\\ &= [m,m]_G(a,b,c)\end{aligned}$$

Definition

A degree 2 Hochschild cohomology class like $[m_1]$ is called a **noncommutative Poisson structure** on A.

When A is $\mathcal{O}_M$, the Hochschild-Kostant-Rosenberg theorem implies that Poisson structures on $\mathcal{O}_M$ are in 1-1 correspondence with Poisson brackets $\{-,-\}$ on $\mathcal{O}_M$.

Questions I

In general,

$$H^2(C^\infty(M)\rtimes\Gamma, C^\infty(M)\rtimes\Gamma)$$
$$= \Gamma(\wedge^2 TM)^\Gamma \oplus \Big(\bigoplus_{\gamma\in\Gamma,\ell(\gamma)=2}\Gamma(\wedge^2 N^\gamma)\Big)^\Gamma.$$

Question

What does the second component in the above expression do to deformations of the algebra $C^\infty(M)\rtimes\Gamma$?

Let M be a symplectic manifold, and Γ acts on M preserving the symplectic structure. Let $(C^\infty(M)[[\hbar]], \star)$ be a Γ invariant star product on M.

Let M be a symplectic manifold, and Γ acts on M preserving the symplectic structure. Let $(C^\infty(M)[[\hbar]], \star)$ be a Γ invariant star product on M.

Theorem (Dolgushev-Etingof, Neumaier-Pflaum-Posthuma-T)

$$\begin{aligned} & H^\bullet(C^\infty(M)((\hbar)) \rtimes \Gamma, C^\infty(M)((\hbar)) \rtimes \Gamma) \\ = \ & H^{\bullet-\ell}(IX, \mathbb{C}((\hbar))). \end{aligned}$$

Hochschild Cohomology of Deformation Quantization

Let M be a symplectic manifold, and Γ acts on M preserving the symplectic structure. Let $(C^\infty(M)[[\hbar]], \star)$ be a Γ invariant star product on M.

Theorem (Dolgushev-Etingof, Neumaier-Pflaum-Posthuma-T)

$$\begin{aligned} & H^\bullet(C^\infty(M)((\hbar)) \rtimes \Gamma, C^\infty(M)((\hbar)) \rtimes \Gamma) \\ = \; & H^{\bullet-\ell}(IX, \mathbb{C}((\hbar))). \end{aligned}$$

Example (Alev-Farinati-Lembre-Solotar)

When M is a symplectic vector space V and Γ acts on V by linear symplectic transformation. Let W be the Weyl algebra on V.

Let M be a symplectic manifold, and Γ acts on M preserving the symplectic structure. Let $(C^\infty(M)[[\hbar]], \star)$ be a Γ invariant star product on M.

Theorem (Dolgushev-Etingof, Neumaier-Pflaum-Posthuma-T)

$$\begin{aligned} & H^\bullet(C^\infty(M)((\hbar)) \rtimes \Gamma, C^\infty(M)((\hbar)) \rtimes \Gamma) \\ = \; & H^{\bullet-\ell}(IX, \mathbb{C}((\hbar))). \end{aligned}$$

Example (Alev-Farinati-Lembre-Solotar)

When M is a symplectic vector space V and Γ acts on V by linear symplectic transformation. Let W be the Weyl algebra on V. The k-th Hochschild cohomology group of $W[\hbar^{-1}] \rtimes \Gamma$ is a vector space over $\mathbb{C}((\hbar))$ with the dimension equal to the number of conjugacy classes of Γ whose codimension is equal to k.

Questions II

Let $C(\gamma)$ be the centralizer group of γ in Γ.

Questions II

Let $C(\gamma)$ be the centralizer group of γ in Γ. The 2nd Hochschild cohomology group of $C^\infty(M)((\hbar)) \rtimes \Gamma$ is equal to

$$H^2(X, \mathbb{C}((\hbar))) \oplus \bigoplus_{<\gamma>, \ell(\gamma)=2} H^0(M^\gamma / C(\gamma), \mathbb{C}((\hbar))).$$

Conjecture (Dolgushev-Etingof)

Deformations of the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ are unobstructed. In particular, the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ has a deformation coming from every γ fixed point component with codimension 2.

Questions II

Let $C(\gamma)$ be the centralizer group of γ in Γ. The 2nd Hochschild cohomology group of $C^\infty(M)((\hbar)) \rtimes \Gamma$ is equal to

$$H^2(X, \mathbb{C}((\hbar))) \oplus \bigoplus_{<\gamma>, \ell(\gamma)=2} H^0(M^\gamma / C(\gamma), \mathbb{C}((\hbar))).$$

Conjecture (Dolgushev-Etingof)

Deformations of the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ are unobstructed. In particular, the algebra $(C^\infty(M)^{\mathbb{Z}_2}((\hbar)), \star)$ has a deformation coming from every γ fixed point component with codimension 2.

Evidences of the above conjecture include : symplectic reflection algebras, cotangent bundles, $\cdots$

Part III : Dunkl operator and a construction for $\mathbb{Z}_2$ orbifolds

In this part, we will discuss some progress toward answering the two questions raised in Part II about formal deformation quantization of orbifold algebras.

Symplectic Reflection Algebra

Let V be a symplectic vector space, and Γ be a finite subgroup of $Sp(V)$. Let π be the corresponding constant Poisson bivector on V.

Let V be a symplectic vector space, and Γ be a finite subgroup of $Sp(V)$. Let π be the corresponding constant Poisson bivector on V.

• Invariant subspace and normal space : For every $\gamma \in \Gamma$, V^γ is a symplectic subspace. Let N^γ be the symplectic orthogonal subspace to V^γ. For γ with $\ell(\gamma) = 2$, define π_γ to be the restriction of π along N^γ.

Symplectic Reflection Algebra

Let V be a symplectic vector space, and Γ be a finite subgroup of $Sp(V)$. Let π be the corresponding constant Poisson bivector on V.

• Invariant subspace and normal space : For every $\gamma \in \Gamma$, V^γ is a symplectic subspace. Let N^γ be the symplectic orthogonal subspace to V^γ. For γ with $\ell(\gamma) = 2$, define π_γ to be the restriction of π along N^γ.

• Noncommutative Poisson structure : The bilinear operator $\Pi = t\pi + \sum_{\gamma, \ell(\gamma)=2} c_\gamma \pi_\gamma U_\gamma$ with $c_{\alpha\gamma\alpha^{-1}} = c_\gamma$ defines a degree 2 Hochschild class on $S(V^*) \rtimes \Gamma$ with $[\Pi, \Pi]_G = 0$.

Symplectic Reflection Algebra

Let V be a symplectic vector space, and Γ be a finite subgroup of $Sp(V)$. Let π be the corresponding constant Poisson bivector on V.

• Invariant subspace and normal space : For every $\gamma \in \Gamma$, V^γ is a symplectic subspace. Let N^γ be the symplectic orthogonal subspace to V^γ. For γ with $\ell(\gamma) = 2$, define π_γ to be the restriction of π along N^γ.

• Noncommutative Poisson structure : The bilinear operator $\Pi = t\pi + \sum_{\gamma, \ell(\gamma)=2} c_\gamma \pi_\gamma U_\gamma$ with $c_{\alpha\gamma\alpha^{-1}} = c_\gamma$ defines a degree 2 Hochschild class on $S(V^*) \rtimes \Gamma$ with $[\Pi, \Pi]_G = 0$.

Theorem (Etingof-Ginzburg)

The algebra $H_{t,c} := T(V^*) \rtimes \Gamma / < xy - yx = \Pi(x, y) >$ is the symplectic reflection algebra introduced by Etingof-Ginzburg, which is a universal deformation of $W \rtimes \Gamma$.

Generalization to Linear Cases

Let $\mathfrak{g}$ be a Lie algebra, and Γ act on $\mathfrak{g}$ by Lie algebra automorphisms. Assume that for every γ, $\ell(\gamma)$ is even.

Generalization to Linear Cases

Let $\mathfrak{g}$ be a Lie algebra, and Γ act on $\mathfrak{g}$ by Lie algebra automorphisms. Assume that for every γ, $\ell(\gamma)$ is even.

• Linear structure : Let V be the dual $\mathfrak{g}^*$, and π be the linear Poisson structure on V associated to the Lie bracket structure on $\mathfrak{g}$. For every $\gamma \in \Gamma$ with $\ell(\gamma) = 2$, let π_γ be the restriction of π on $V^{*\gamma} \otimes N^\gamma \wedge N^\gamma$.

Let $\mathfrak{g}$ be a Lie algebra, and Γ act on $\mathfrak{g}$ by Lie algebra automorphisms. Assume that for every γ, $\ell(\gamma)$ is even.

- Linear structure : Let V be the dual $\mathfrak{g}^*$, and π be the linear Poisson structure on V associated to the Lie bracket structure on $\mathfrak{g}$. For every $\gamma \in \Gamma$ with $\ell(\gamma) = 2$, let π_γ be the restriction of π on $V^{*\gamma} \otimes N^\gamma \wedge N^\gamma$.
- Noncommutative Poisson structure : The bilinear operator $\Pi = t\pi + \sum_{\gamma, \ell(\gamma)=2} c_\gamma \pi_\gamma U_\gamma$ with $c_{\alpha\gamma\alpha^{-1}} = c_\gamma$ for any α defines a degree 2 Hochschild cohomology class on $S(V^*) \rtimes \Gamma$ with $[\Pi, \Pi]_G = 0$.

Theorem (Halbout-Oudom-T)

Associated to such a bilinear operator Π, one can construct a deformation of the algebra $U\mathfrak{g} \rtimes \Gamma$.

A particular type of Poisson structure

Let M be a symplectic manifold, and Γ act on M by symplectic diffeomorphisms.

Let M be a symplectic manifold, and Γ act on M by symplectic diffeomorphisms.

Proposition

Let π be the Poisson structure associated to the symplectic form on M, and π_γ is the restriction of π to the normal bundle of M^γ with $\ell(\gamma) = 2$. Then $\Pi = \pi + \sum_{\gamma, \ell(\gamma)=2} c_\gamma \pi_\gamma U_\gamma$ for $c_\gamma = c_{\alpha\gamma\alpha^{-1}}$ defines a noncommutative Poisson structure on $C^\infty(M) \rtimes \Gamma$.

A particular type of Poisson structure

Let M be a symplectic manifold, and Γ act on M by symplectic diffeomorphisms.

Proposition

Let π be the Poisson structure associated to the symplectic form on M, and π_γ is the restriction of π to the normal bundle of M^γ with $\ell(\gamma) = 2$. Then $\Pi = \pi + \sum_{\gamma, \ell(\gamma)=2} c_\gamma \pi_\gamma U_\gamma$ for $c_\gamma = c_{\alpha\gamma\alpha^{-1}}$ defines a noncommutative Poisson structure on $C^\infty(M) \rtimes \Gamma$.

In the next part, we discuss how to quantize such a Poisson structure Π in the case that $\Gamma = \mathbb{Z}_2$.

Dunkl operator

We can define a bilinear operator $\Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^2)$ by

$$\Delta(f)(x,y) = \frac{f(x) - f(y)}{x - y}.$$

Dunkl operator

We can define a bilinear operator $\Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^2)$ by

$$\Delta(f)(x,y) = \frac{f(x) - f(y)}{x - y}.$$

Proposition

1. $\Delta(f)(x,x) = D(f)(x) = \frac{df}{dx}(x)$,
   $\Delta(f)(x,-x) = \tilde{D}(f)(x) = \frac{f(x)-f(-x)}{2x}$.
2. Δ is coassociative and cocommutative.
3. $\Delta(f) = (f \otimes 1)\Delta(g) + \Delta(f)(1 \otimes g)$.

Define the **Dunkl operator** to be

$$T_k(f)(x) = \frac{df}{dx}(x) + k\frac{f(x) - f(-x)}{x} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}).$$

Operator product

We the following two sets of linear operator on $C^\infty(\mathbb{R})$.

Operator product

We the following two sets of linear operator on $C^\infty(\mathbb{R})$.

1. The operator $\mathrm{Op}_k(x)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(x)(f)(x) = xf(x)$.

Operator product

We the following two sets of linear operator on $C^\infty(\mathbb{R})$.

1. The operator $\mathrm{Op}_k(x)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(x)(f)(x) = xf(x)$.
2. The operator $\mathrm{Op}_k(p)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(p)(f)(x) = -iT_k(f)(x)$.

Operator product

We the following two sets of linear operator on $C^\infty(\mathbb{R})$.

1. The operator $\mathrm{Op}_k(x)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(x)(f)(x) = xf(x)$.
2. The operator $\mathrm{Op}_k(p)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(p)(f)(x) = -iT_k(f)(x)$.

The commutator $[\mathrm{Op}_k(p), \mathrm{Op}_k(x)]$ is equal to $-i(1+2k\hat{e})$.

Operator product

We the following two sets of linear operator on $C^\infty(\mathbb{R})$.

1. The operator $\mathrm{Op}_k(x)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(x)(f)(x) = xf(x)$.
2. The operator $\mathrm{Op}_k(p)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(p)(f)(x) = -iT_k(f)(x)$.

The commutator $[\mathrm{Op}_k(p), \mathrm{Op}_k(x)]$ is equal to $-i(1+2k\hat{e})$.
In general, $\mathrm{Op}_k(a_1) \circ \mathrm{Op}_k(a_2)$ has the following form.

Operator product

We the following two sets of linear operator on $C^\infty(\mathbb{R})$.

1. The operator $\mathrm{Op}_k(x)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(x)(f)(x) = xf(x)$.
2. The operator $\mathrm{Op}_k(p)$ acts on $C^\infty(\mathbb{R})$ by $\mathrm{Op}_k(p)(f)(x) = -iT_k(f)(x)$.

The commutator $[\mathrm{Op}_k(p), \mathrm{Op}_k(x)]$ is equal to $-i(1+2k\hat{e})$.
In general, $\mathrm{Op}_k(a_1) \circ \mathrm{Op}_k(a_2)$ has the following form.

$$\sum_{j,l} k^l \Big(\mathrm{Op}_k \left(C^0_{j,l}(a_1, a_2)\right) + \mathrm{Op}_k \left(C^1_{j,l}(a_1, a_2)\right) \circ \hat{\gamma} \Big).$$

A "Moyal" type formula

Definition

Define an associative product $\star$ on $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]]$ by

A "Moyal" type formula

Definition

Define an associative product $\star$ on $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]]$ by

1. $\star$ is $\mathbb{C}[[\hbar_1, \hbar_2]]$ linear ;

A "Moyal" type formula

Definition

Define an associative product $\star$ on $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]]$ by

1. $\star$ is $\mathbb{C}[[\hbar_1, \hbar_2]]$ linear ;
2. For $a_1, a_2 \in C^\infty(\mathbb{R}^2)$, $a_1 \star a_2$ is defined by

$$a_1 \star a_2 = \sum_{j,l} \hbar_1^j \hbar_2^l (C^0_{j,l}(a_1, a_2) + C^1_{j,l}(a_1, a_2) U_\gamma).$$

A "Moyal" type formula

Definition

Define an associative product $\star$ on $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]]$ by

1. $\star$ is $\mathbb{C}[[\hbar_1, \hbar_2]]$ linear;
2. For $a_1, a_2 \in C^\infty(\mathbb{R}^2)$, $a_1 \star a_2$ is defined by

$$a_1 \star a_2 = \sum_{j,l} \hbar_1^j \hbar_2^l (C^0_{j,l}(a_1, a_2) + C^1_{j,l}(a_1, a_2) U_\gamma).$$

This algebra $(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]], \star)$ is called the "Dunkl-Weyl" algebra.

Definition

Define an associative product $\star$ on $C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]]$ by

1. $\star$ is $\mathbb{C}[[\hbar_1, \hbar_2]]$ linear ;
2. For $a_1, a_2 \in C^\infty(\mathbb{R}^2)$, $a_1 \star a_2$ is defined by

$$a_1 \star a_2 = \sum_{j,l} \hbar_1^j \hbar_2^l (C^0_{j,l}(a_1, a_2) + C^1_{j,l}(a_1, a_2) U_\gamma).$$

This algebra $(C^\infty(\mathbb{R}^2) \rtimes \mathbb{Z}_2[[\hbar_1, \hbar_2]], \star)$ is called the "Dunkl-Weyl" algebra.

When $\hbar_2 = 0$, $a_1 \star a_2 = \sum_{j=0}^\infty \frac{(-i)^j \hbar_1^j}{j!} \partial_p^j(a_1) \partial_x^j(a_2)$.

Theorem (Halbout-T)

Let M be a symplectic manifold, and $\mathbb{Z}_2$ act on M symplectically.

Quantization of $\mathbb{Z}_2$-orbifolds

Theorem (Halbout-T)

Let M be a symplectic manifold, and $\mathbb{Z}_2$ act on M symplectically. Let $(C^\infty(M)[[\hbar]], \star)$ be a $\mathbb{Z}_2$-invariant star product on M with the characteristic class being $-\omega$.

Quantization of $\mathbb{Z}_2$-orbifolds

Theorem (Halbout-T)

Let M be a symplectic manifold, and $\mathbb{Z}_2$ act on M symplectically. Let $(C^\infty(M)[[\hbar]], \star)$ be a $\mathbb{Z}_2$-invariant star product on M with the characteristic class being $-\omega$.

The invariant algebra $(C^\infty(M)((\hbar))^{\mathbb{Z}_2}, \star)$ has a deformation corresponding to every codimension 2 fixed point submanifold.

Quantization of $\mathbb{Z}_2$-orbifolds

Theorem (Halbout-T)

Let M be a symplectic manifold, and $\mathbb{Z}_2$ act on M symplectically. Let $(C^\infty(M)[[\hbar]], \star)$ be a $\mathbb{Z}_2$-invariant star product on M with the characteristic class being $-\omega$.

The invariant algebra $(C^\infty(M)((\hbar))^{\mathbb{Z}_2}, \star)$ has a deformation corresponding to every codimension 2 fixed point submanifold.

In a joint work with Ramadoss, we studied the cyclic cohomology and local algebraic index theory on the deformation constructed in the above theorem.

Outlook :

1. Many of our constructions and results rely heavily on the assumption that we are working with $\mathbb{R}$ and $\mathbb{C}$. Many questions we are discussing today have natural generalizations to an arbitrary field.

Outlook :

1. Many of our constructions and results rely heavily on the assumption that we are working with $\mathbb{R}$ and $\mathbb{C}$. Many questions we are discussing today have natural generalizations to an arbitrary field.
2. Generalize the $\mathbb{Z}_2$ result to general cyclic groups.

Outlook :

1. Many of our constructions and results rely heavily on the assumption that we are working with $\mathbb{R}$ and $\mathbb{C}$. Many questions we are discussing today have natural generalizations to an arbitrary field.
2. Generalize the $\mathbb{Z}_2$ result to general cyclic groups.
3. In the $\mathbb{Z}_2$ case, we have only considered a special type of Poisson structure naturally from the symplectic form.

Outlook :

1. Many of our constructions and results rely heavily on the assumption that we are working with $\mathbb{R}$ and $\mathbb{C}$. Many questions we are discussing today have natural generalizations to an arbitrary field.
2. Generalize the $\mathbb{Z}_2$ result to general cyclic groups.
3. In the $\mathbb{Z}_2$ case, we have only considered a special type of Poisson structure naturally from the symplectic form.
4. Sharapov and Skvortsov recently discovered an interesting connection to high spin gravity.

Outlook :

1. Many of our constructions and results rely heavily on the assumption that we are working with $\mathbb{R}$ and $\mathbb{C}$. Many questions we are discussing today have natural generalizations to an arbitrary field.
2. Generalize the $\mathbb{Z}_2$ result to general cyclic groups.
3. In the $\mathbb{Z}_2$ case, we have only considered a special type of Poisson structure naturally from the symplectic form.
4. Sharapov and Skvortsov recently discovered an interesting connection to high spin gravity.
5. Construct a right sigma model to solve the quantization problem.

Thank you for your attention!