Introduction to higher-spin gravity (for mathematicians)

Xavier Bekaert

Institut Denis Poisson
Université François Rabelais de Tours

17 February 2020 @ ULB & Solvay Institutes

Higher-spin gravity : definition and main challenge

Higher-spin gravity

- $\equiv$ interacting relativistic field theory whose spectrum contains a tower of massless fields with at least:
  - one massless field of spin two (the "graviton"),
  - one massless field of spin greater than two ("higher spin").
- Notoriously difficult to introduce consistent interactions between higher-spin fields (especially in dimensions strictly greater than 3).
  - In particular, the spectrum must necessarily contain an infinite tower of fields of unbounded spin (under mild hypotheses).
  - Another obstacle is that there is a tension between higher-spin gauge symmetries and minimal coupling to gravity (especially around flat spacetime and assuming parity symmetry).

"Popsci" of higher-spins: elementary particles & symmetries

Standard model

Particle	Field	Spin
Brout-Englert-Higgs	Scalar	0
Leptons & Quarks	Spinor	1/2
Gauge bosons	Vector	1

Underlying "gauged" symmetries: internal symmetries

(Ehresmann connection on a principal bundle with structure group which is a compact semi-simple Lie group)

"Popsci" of higher-spins: elementary particles & symmetries

Standard model + Gravity

Particle	Field	Spin
Brout-Englert-Higgs	Scalar	0
Leptons & Quarks	Spinor	1/2
Gauge bosons	Vector	1
Graviton (?)	Tensor	2

Extra "gauged" symmetries: spacetime isometries

(Cartan connection on a principal bundle of tangent frames with an isometry group (of a maximally-symmetric Lorentzian manifold) as structure group)

"Popsci" of higher-spins: elementary particles & symmetries

Standard model + (Super)gravity

Particle	Field	Spin
Higgs	Scalar	0
Leptons & Quarks	Spinor	1/2
Gauge bosons	Vector	1
Gravitino (??)	Spin-vector	3/2
Graviton (?)	Tensor	2

Extra "gauged" symmetries: supersymmetry

(Supergeometry)

"Popsci" of higher-spins: elementary particles & symmetries

Higher-spin (super)gravity

Particle	Field	Spin
Higgs	Scalar	0
Leptons & Quarks	Spinor	1/2
Gauge bosons	Vector	1
Gravitino (??)	Spin-vector	3/2
Graviton (?)	Tensor	2
Higher-spin (???)	Spin-tensor	s>2

Extra "gauged" symmetries: higher-derivative symmetries

Outline

1 Introduction

2 (Pre)history of higher-spin fields
- Prehistory
- Modern times

3 Mathematics of higher-spin gravity: some basic ingredients
- Higher-spin gauge fields and symmetries
- Feffermann-Graham-like holography
- Cartan-like geometry

Very short history of higher-spin particles (early days)

Slicing (pre)history

- **1932-1939**: "Birth: generalise Dirac"
- **1939 et 40's**: "Foundations: elementary particles as unitary irreducible representations"
- **decades 50-60**: "The demographic explosion: the hadronic boom"
- **70's**: "The Lagrangian quest"
- **1978**: "A well posed problem: the Frønsdal programme"

Prehistory of higher-spin fields

Well posed problems

Early theoretical works on higher-spins can be structured around 4 main questions, formulated here in terms of mathematical classification research programs (ordered in logical progression).

- **Wigner's programme (1939):**
  Unitary representations of isometry groups
- **Bargmann-Wigner's programme (1948):**
  Relativistic wave equations
- **Fierz-Pauli's programme (1939):**
  Variational principles
  (inverse problem of variational calculus)
- **Frønsdal's programme (1978):**
  Consistent interactions
  (Noether method)

1932-1939 : "Birth: generalise Dirac"

Chronology (retrospective view) of linear relativistic wave equations describing a free quantum particle:

- **masseless, spin 0:** d'Alembert (1747)
- **masseless, spin 1:** Maxwell (1873)
- **spin 0:** Klein-Gordon (1926)
- **spin 1/2:** Dirac (1928)
- **arbitrary (half)integer spin:** Majorana (1932) and Dirac (1936)

1932-1939 : “Birth: generalise Dirac”

$\Rightarrow$ Large zoo of relativistic equations

How to put some order (i.e. classify inequivalent ones) ?

1939 and 40's: "Foundations: elementary particles as unitary irreducible representations"

In 1939, Wigner offered his extremely profound and modern view on linear relativistic wave equations.

Combining the axioms of quantum mechanics and the principles of special relativity necessarily leads to the following identifications:

Wave equation describing a free quantum relativistic particle
$\Updownarrow$
Unitary representation of the spacetime isometry group

Space of states (rays) of a free quantum relativistic particle
$\Updownarrow$
Unitary module of the spacetime isometry group

1939 and 40's: "Foundations: elementary particles as unitary irreducible representations"

This identification between free particles and linear representation is perfect on maximally-symmetric spacetimes (= Riemannian manifolds with Lorentz signature and constant curvature):

- **Minkowski** $\mathbb{R}^{D-1,1}$: zero scalar curvature

  Poincaré group $ISO(D-1,1) := \mathbb{R}^{D-1,1} \rtimes SO(D-1,1)$

- **de Sitter** dS_D: constant positive scalar curvature

  Pseudo-orthogonal group $SO(D,1)$ *of Lorentzian signature*

- **anti de Sitter** AdS_D: constant negative scalar curvature

  Pseudo-orthogonal group $SO(D-1,2)$ *of conformal signature*

1939 and 40's: "Foundations: elementary particles as unitary irreducible representations"

These identifications motivate the following mathematical problem

Wigner's programme: *Classify all unitary irreducible representations of the isometry groups of maximally-symmetric spacetimes*

This programme somewhat gave birth to the modern theory of representations by the subsequent works of mathematical physicists such as Bargmann, Gel'fand, Harish-Chandra, ...

1939 and 40's: "Foundations: elementary particles as unitary irreducible representations"

The next step consists in going from the abstract representation to a more concrete realisation (as solution space of a PDE). This step is not trivial because it is not algorithmic: writing relativistic equations is some sort of art.

Bargmann-Wigner's programme*: Associate a linear covariant differential equation to each unitary irreducible representation of the isometry group of maximally-symmetric spacetimes, such that the space of inequivalent solutions carries the corresponding representation.*

1939 and 40's: "Foundations: elementary particles as unitary irreducible representations"

Let us fix ideas with a "concrete" example: the modules of the Poincaré group describing the propagation of free massive particles on Minkowski spacetime can be realised as spaces of tensor fields $\varphi_{\mu_1\dots\mu_r}(x)\in\Gamma(\otimes T^*\mathbb{R}^{D-1,1})$ which are

- solutions of Klein-Gordon equation

$$(\Box - m^2)\,\varphi_{\mu_1\dots\mu_r}(x) = 0\,,$$

  where $\Box$ is the d'Alembertian and $m>0$ is the mass,
- divergenceless

$$\partial^\nu\varphi_{\mu_1\dots\,\nu\dots\mu_r}(x) = 0\,.$$

Moreover, in order to have irreducibility under the Poincaré group $ISO(D-1,1)$, these fields must take value in an irreducible representation of the Lorentz subgroup $SO(D-1,1)$.

Decades 50-60: "The demographic explosion: the hadronic boom"

Evolution of the number of distinct particles observed experimentally:

- **Middle of 40's:** can count on the fingers of one hand (electron, photon, proton, neutron, muon)
- **Beginning of 70's:** more than 50 (today: more than 150)

However, most of these particles are hadrons. Such particles are not elementary but composite (pairs or triplets of quarks).

At the beginning of the 60's, the proliferation of hadrons with "high" ($\geqslant 3/2$) spin was one of the main mystery of the strong nuclear interaction. Plots suggested the existence of an infinite tower of hadrons, with unbounded spin.

70's: "The Lagrangian quest"

The attempts to model scattering cross sections of higher-spin hadrons required the knowledge of the propagators for fields of arbitrary spin. This provided a new motivation for

Fierz-Pauli's programme*: Associate a quadratic local covariant Lagrangian to each unitary irreducible representation of the isometry group of maximally-symmetric spacetimes, such that the space of inequivalent solutions to Euler-Lagrange equations carries the corresponding representation.*

70's: "The Lagrangian quest"

Completion of Fierz-Pauli's programme in Minkowski spacetime of dimension $D = 4$ for the representations

- **massive (arbitrary spin):** Singh & Hagen (1974)
- **massless (arbitrary helicity):** Fang & Frønsdal (1978)

Massless particles

The propagation of free massless integer-spin particles on Minkowski spacetime are described by tensor fields $\varphi_{\mu_1\dots\mu_r}(x)$ on $\mathbb{R}^{D-1,1}$ which are harmonic, $\Box\,\varphi_{\mu_1\dots\mu_r}(x)=0$ and obey to various supplementary conditions.

With respect to the massive case, another novelty is the existence of gauge symmetries (equivalence relations)

$$\varphi_{\mu_1\dots\mu_r}(x)\,\sim\,\varphi_{\mu_1\dots\mu_r}(x)\,+\,\partial_{\mu_1}\varepsilon_{\mu_2\dots\mu_r}(x)\,+\,\cdots$$

Massless particles

Their non-Abelian deformations for “low” spins are well known:

Spin	Theory	Geometry	Field	Symmetries
1	Yang-Mills	Principal bundles	Connection	Internal
2	Gravitation	Pseudo-Riemannian	Metric	Diffeos

Gupta’s programme: Non-geometrical reconstruction of the Yang-Mills and Einstein theories via the perturbative introduction of consistent interactions to the quadratic Lagrangians.

1978: “A well posed problem: the Frønsdal programme”

Two recent achievements for massless particles

- **<u>1976</u>:** generalisation of Gupta’s programme for all “low” spins ($s \leqslant 2$) via supergravity
- **<u>1978</u>:** completion of Fierz-Pauli’s programme for all “higher” spins

lead Frønsdal to further generalise Gupta’s programme for arbitrary spins.

1978: "A well posed problem: the Frønsdal programme"

Frønsdal programme*: List of all interactions*

- ***perturbative****,*
- ***consistent****,*
- ***covariant****,*
- ***local****,*
- ***deforming*** *a positive sum (finite or not) of quadratic (local covariant) Lagrangians associated with unitary irreducible representations of the isometry group of a maximally symmetric spacetime,*
- ***non Abelian****, i.e. such that the algebra of gauge symmetries is non commutative already at first order in the deformation parameter(s).*

Modern times: no-go vs yes-go

Despite an intimidating list of no-go theorems[1] seminal yes-go results were obtained on cubic vertices during the eighties, culminating with the identification of several key ingredients[2] of higher-spin gravity around AdS_4 spacetime by Fradkin and Vasiliev. These encouraging results on the existence of higher-spin gravity were further supported in the early nineties when Vasiliev proposed his fully nonlinear equations.

[1] such as Coleman-Mandula no-go theorem, Weinberg low-energy theorems, Aragone-Deser obstruction, ...

[2] in particular: higher-spin algebras, frame-like formulation, actions à la MacDowell-Mansouri consistent till cubic order, ...

Modern times: Strings

The theoretical study of hadronic physics gave birth to string theory, the spectrum of which is made of an infinite pyramid of particles with unbounded spin.

All particules in exotic representations (higher spin, mixed symmetries) have a mass above (or of the order of) Planck mass ($\approx 10^{19}$ proton mass).

This infinite pyramid of extremely massive higher-spin particles is responsible for the very good ultraviolet behaviour (UV finiteness) of string theory.

Modern times: Strings

$\Rightarrow$ From the point of view of Frønsdal's programme, string field theory is a highly nontrivial example of consistent interacting theory of *massive* higher-spin particles.

$\Leftarrow$ Conversely, the development of Frønsdal's programme could shed new light on

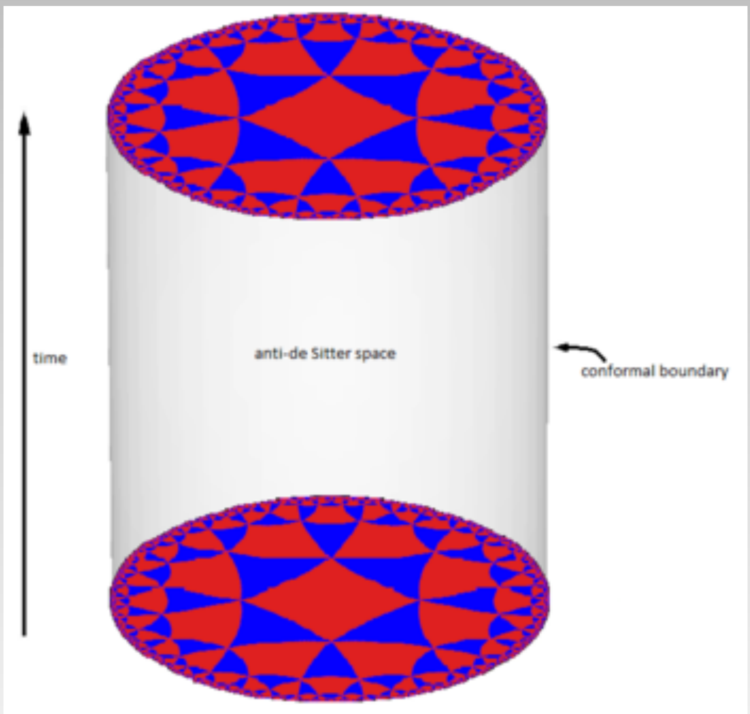
- string theory:
  ultraviolet behaviour, underlying symmetry principle, ...
- AdS/CFT correspondence:
  holographic duality in the regime of
  strong curvature $\leftrightarrow$ weak coupling, ...

Modern times: Spacetime reconstruction

At the beginning of this century, a major conceptual change of perspective on higher-spin gravity and its potential relevance for high-energy physics was brought by holographic duality (aka the "AdS/CFT correspondence").

Modern times: spacetime reconstruction

A **holographic duality** is an equivalence between a theory of quantum gravity in the bulk of (an asymptotically) AdS spacetime $\mathcal{M}$ and a CFT (without gravity) living on the conformal boundary $\partial\mathcal{M}$.



Modern times: spacetime reconstruction

The semiclassical limit (tree approximation) in the bulk gravitational theory corresponds to the limit of a large number of fields on the boundary.

Usually the bulk is weakly curved and corresponds to a strongly coupled CFT on the boundary, but the converse limit is also of interest in the sense that in principle one should be able to reconstruct perturbatively the gravitational theory in the strongly curved bulk from the weakly coupled CFT.

In particular, *integrable CFTs should be dual to unbroken higher-spin gravity theories in the bulk*.

One of the most inspiring example of such holographic duality is that: *Higher-spin gravity around AdS_4 is dual to the Wilson-Fisher fixed point of the $O(N)$ model.*

Mathematics of higher-spin gravity: some basic ingredients

- Higher-spin gauge fields
  - metric-like and frame-like formulations
  - two main strategies for introducing interactions
- Higher-spin holography à la Feffermann-Graham
  - Flat ambient model (Möbius, Dirac)
  - Curved version (Feffermann-Graham)
  - Higher-order generalisation
- Higher-spin generalisations of gravity via Cartan-like geometry
- Jet prolongation and unfolding

Higher-spin gauge symmetries

Metric-like formulation: higher-spin gauge fields as totally-symmetric covariant tensor fields $\varphi_{\mu_1\cdots\mu_s}(x) \in \Gamma(\odot^s T^* M)$ of higher rank $s > 2$.

Spin	Gauge field	Gauge symmetry
1	Vector gauge field	$\delta A_\mu = \partial_\mu \Lambda$
2	Metric tensor field	$\delta g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$
3, ...	Higher-spin gauge fields	$\delta\varphi_{\mu\nu\rho} = \nabla_{(\mu}\xi_{\nu\rho)} + \mathcal{O}(\varphi)$ ...

Higher-spin gauge symmetries

The geometries underlying those non Abelian gauge symmetries are well known for "low" spins but remain somewhat elusive for "higher" spins.

Spin	Theory	Geometry	Field	Lie algebra
1	Yang-Mills	Principal bundles	Connection	internal
2	Gravitation	Riemannian	Metric	vector fields
$3, \ldots$	Higher-spin	??	??	differential operators

Higher-spin gauge symmetries

The structure of non Abelian symmetries generically require, for consistency, an infinite tower of gauge fields with unbounded spin. In the metric-like formulation, they are conveniently packed into a single generating function

$$\varphi(x,p) = \sum_s \varphi^{\mu_1\cdots\mu_s}(x)\, p_{\mu_1}\cdots p_{\mu_s} \in \Gamma(\odot TM) \subset C^\infty(T^*M)\,.$$

Remark: Indices have been raised via the background metric in order to already suggest their suggestive interpretation as symbols of differential operators.

Higher-spin interactions

The two main paths for addressing the interaction problem are based either on the (equivalent)

- **metric-like formulation**
- **frame-like formulation**

Higher-spin interactions

The two main paths for addressing the interaction problem are based either on the (equivalent)

- **metric-like formulation:**
  - *systematic classification of vertices*, consistent with gauge symmetries and modulo field redefinitions,
    - as a cohomological problem (which can be handled via various mathematical technologies, such as the local BRST cohomology).
  - *(re)construction of vertices* and/or study of their physical properties (such as locality, etc)
    - either as scattering amplitudes ($\Rightarrow$ on-shell vs off-shell subtleties, etc) in flat or AdS spacetimes,
    - or as effective action vertices (in the case of conformal higher-spin gravity).
- **frame-like formulation**

Higher-spin interactions

The two main paths for addressing the interaction problem are based either on the (equivalent)

- **metric-like formulation**
- **frame-like formulation:**

  Construct fully interacting theories by looking for inspiration in Cartan's

  - *view of gravity* (Cartan connection on a principal bundle)

    $\Rightarrow$ Replace the isometry algebra by a higher-spin extension.

  - *method of prolongation* (Geometry of PDEs)

    $\Rightarrow$ Reformulate the equations of motion as an exterior differential system (or its modern avatars such as L_∞ algebras, etc).

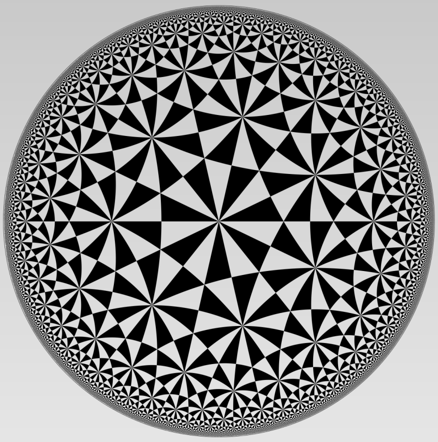
Higher-spin algebra

Higher-spin gravity theories are, to a large extent, determined by their higher-spin algebra which are best understood as the algebra of symmetries of a free conformal field living on the boundary.

Möbius model

Consider the conformal sphere $S^d \cong \mathbb{R}^d \cup \infty$ which can be obtained as the compactification of the Euclidean space $\mathbb{R}^d$.

It is the conformal boundary of the hyperbolic (aka "Euclidean AdS") space $EAdS_{d+1}$.

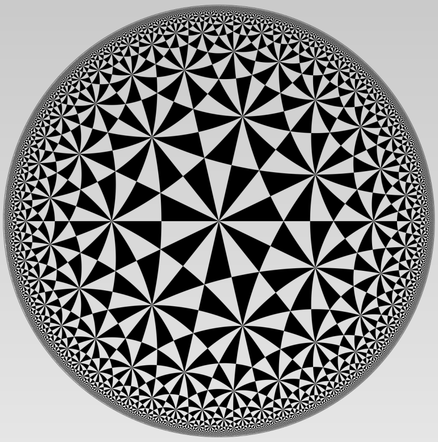


Tiling on the Poincaré disk (Wikimedia Commons)

Möbius model

The group $G = O(d+1,1)$ of conformal isometries of S^d is generated by translations, rotations, dilatations and special conformal transformations.

The subgroup $H \subset G$ fixing a point of the conformal sphere $S^d \cong G/H$ is a parabolic subgroup generated by rotations, dilatations and special conformal transformations.

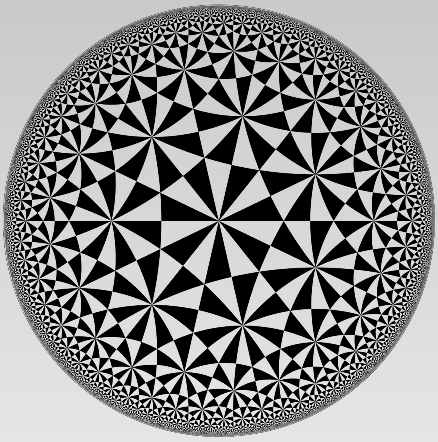


Tiling on the Poincaré disk (Wikimedia Commons)

Möbius model

The same group $G = O(d+1,1)$ identifies with the group of isometries of the hyperbolic space $EAdS_{d+1}$.

However, the orthogonal group $O(d+1) \subset G$ is the isotropy subgroup a point of $EAdS_{d+1} \cong O(d+1,1)/O(d+1)$.

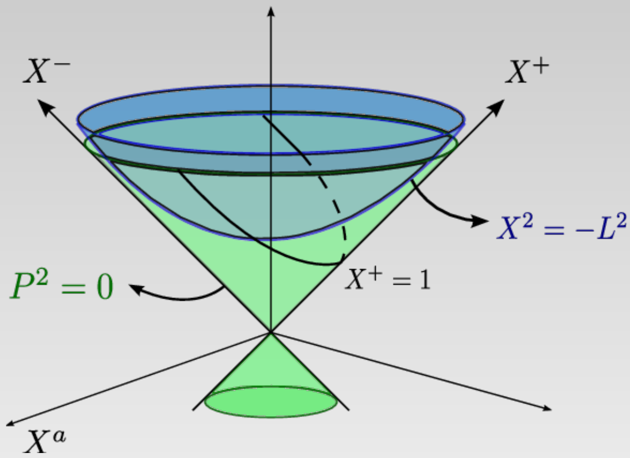


Tiling on the Poincaré disk (Wikimedia Commons)

Möbius model

These symmetries are made manifest by considering the (upper) null cone $\mathcal{N} \subset \mathbb{R}^{d+1,1}$ inside an ambient Minkowski spacetime.

The conformal sphere S^d (respectively, the hyperbolic space $EAdS_{d+1}$) is identified with the space of null rays (respectively, one sheet of the two-sheeted hyperboloid).



Dirac hypercone

The conformal compactification M_d of the Minkowski spacetime is the conformal boundary of the anti de Sitter spacetime AdS_{d+1}.

The group $O(d,2)$ of conformal isometries is made manifest by considering the null cone $\mathcal{N} \subset \mathbb{R}^{d,2}$.

The space M_d (and AdS_{d+1}) are identified with the space of null rays (and the hyperboloid).

Conformal scalar field

The group-theoretical definition of a free on-shell conformal scalar field with canonical dimension is:

Scalar singleton: The space $\mathcal{D}(\frac{d}{2}-1,0)$ of solutions of the d'Alembert equation in d dimensions,

$$\Box\phi(x)=0$$

where $\Box$ is the d'Alembertian and $\phi(x)$ has conformal weight $\frac{d}{2}-1$, carries a unitary irreducible representation of the conformal algebra $\mathfrak{so}(d,2)$ (sometimes called the minimal representation), whose generators are realized as first-order differential operators (e.g. $x\cdot\partial+\frac{d}{2}-1$ for dilatations).

Conformal scalar field in conformally flat background

Equivalently, this space of solution of the d'Alembert equation in d dimensions,

$$\Box\phi(x) = 0$$

can be realised as the space of (equivalence classes of) functions $\Phi(X) \in C^\infty(\mathbb{R}^{d,2})$ on the ambient space, which are

1. harmonic, $\Box\Phi(X) = 0$,
2. homogeneous, $(X \cdot \partial + \frac{d}{2} - 1)\Phi(X) = 0$,
3. defined modulo any function vanishing on the null cone, $\Phi(X) \sim \Phi(X) + X^2\alpha(X)$.

These three conditions can be obtained via first-quantisation of a particle model with the 3 first-class constraints P^2, $X \cdot P$, X^2.
The underlying Lie algebra spanned by their Poisson bracket is $\mathfrak{sp}(2, \mathbb{R})$.

Higher-spin algebra

Definition 1: *The type-A higher-spin algebra* $\mathfrak{hs}(d,2)$ *is the algebra of infinitesimal symmetry generators of the wave equation* $\Box\phi = 0$, *i.e. differential operators* $\hat{A}$ *such that*

$$\Box\hat{A} = \hat{B}\,\Box$$

and modulo trivial generators

$$\hat{A} \sim \hat{A} + \hat{C}\,\Box\,.$$

The above infinitesimal symmetry generators span an associative algebra. Stricty speaking, the higher-spin algebra $\mathfrak{hs}(d,2)$ is (a real form) of the corresponding Lie algebra endowed with commutator as Lie bracket.

Higher-spin algebra

Definition 2: *The higher-spin algebra* $\mathfrak{hs}(d,2)$ *is the enveloping algebra of* $\mathfrak{so}(d,2)$ *generators, quotiented by the ideal corresponding to the wave operator* $\Box$.

Higher-spin algebra

Definition 2': *The higher-spin algebra is the quotient of the universal enveloping algebra of* $\mathfrak{so}(d,2)$ *by the Joseph ideal, which is the annihilator of the singleton module* $\mathcal{D}(\frac{d}{2}-1,0)$,

$$\mathfrak{hs}(d,2) \cong \mathcal{U}\Big(\,\mathfrak{so}(d,2)\,\Big)\,/\,Ann\,\mathcal{D}(\frac{d}{2}-1,0)$$

Higher-spin algebra

The Lie algebras $\mathfrak{sp}(2,\mathbb{R})$ and $\mathfrak{so}(d,2)$ form a Howe dual pair inside $\mathfrak{sp}\big(\,2(d+2)\,\big)$, in the sense that they centralise each other.

Definition 3: *The higher-spin algebra is the centraliser of $\mathfrak{sp}(2,\mathbb{R})$ inside the Weyl algebra $A_{2(d+2)}$ quotiented by the ideal generated by elements proportional to $\mathfrak{sp}(2,\mathbb{R})$ elements*

$$\mathfrak{hs}(d,2) \cong Z_{A_{2(d+2)}}\Big(\,\mathfrak{sp}(2,\mathbb{R})\,\Big)\,/\,\langle\mathfrak{sp}(2,\mathbb{R})\rangle$$

Remark: This definition motivates the use of Moyal star-product on $T^*\mathbb{R}^{d+2}$ in concrete computations.

Higher-spin algebra

Other equivalent definitions of the higher-spin algebra exist and some simpler convenient realisations via star-products are available in particular spacetime dimensions.

Ambient metric

Fefferman-Graham introduced a curved version of the Möbius model where a d-dimensional conformal manifold is

- the base space of a Carrollian manifold (i.e. a principal $\mathbb{R}$-bundle $\mathcal{N}$ endowed with a degenerate metric whose kernel is the fundamental vector field) whose sections are the Riemannian metric representing the conformal class.
- (formally) the conformal boundary of a $(d+1)$-dimensional Einstein manifold.

Ambient metric

Moreover, there exists (formally) a $(d+2)$-dimensional Ricci-flat manifold $\mathcal{M}_{d+2}$ which is a principal $\mathbb{R}$-bundle such that

1. its metric is of homogeneity degree 2 with respect to the fundamental vector field, $\mathcal{L}_V G = 2G$
2. the one-form $A = G(V, \cdot)$ dual to the fundamental vector field is closed, $dA = 0$.

In even dimension, the (formal) existence of this ambient metric requires the d-dimensional conformal manifold to be Bach-flat (i.e. conformal gravity must be on-shell).

Ambient metric

Note that, one may relax the Ricci-flatness of the ambient metric (but keep the same two conditions) and any d-dimensional conformal manifold is

- the base space of a Carrollian manifold (i.e. a principal $\mathbb{R}$-bundle $\mathcal{N}$ endowed with a degenerate metric whose kernel is the fundamental vector field) whose sections are the Riemannian metric representing the conformal class.
- (formally) the conformal boundary of a $(d+1)$-dimensional Riemannian manifold.

Conformal scalar field in curved background

A crucial observation of GJMS is that the space of solution of the Yamabe equation in d dimensions,

$$\Big(\nabla^2 - \tfrac{d-2}{4(d-1)}\mathcal{R}\Big)\phi(x) = 0$$

can be realised as the space of (equivalence classes of) functions $\Phi(X) \in C^\infty(\mathcal{M}_{d+2})$ which are

1. harmonic, $\Delta\Phi(X) = 0$,
2. homogeneous, $\big(V(X)\cdot\partial + \frac{d}{2} - 1\big)\Phi(X) = 0$,
3. defined modulo any function vanishing on the null submanifold, $\Phi(X) \sim \Phi(X) + V^2(X)\alpha(X)$,

where the above 3 differential operators (of respective order 2, 1, 0) span the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ like in the flat case.

Conformal scalar field in curved background

In fact, relaxing the Ricci-flatness, the two conditions on the ambient metric are equivalent to the fact that the 3 differential operators span the Lie algebras $\mathfrak{sp}(2,\mathbb{R})$.

Conformal scalar field in higher-spin background

A somewhat natural generalisation is to consider the space of solutions of the Yamabe equation in d dimensions deformed by a higher-order Hermitian differential operator,

$$(\nabla^2 - \tfrac{d-2}{4(d-1)}\mathcal{R} + \hat{h})\phi(x) = 0\,.$$

This space can be realised as the space of (equivalence classes of) functions $\Phi(X) \in C^\infty(\mathcal{M}_{d+2})$ which are

1. harmonic, $(\Delta + \hat{H})\Phi(X) = 0$,
2. homogeneous, $\big(V(X)\cdot\partial + \frac{d}{2} - 1\big)\Phi(X) = 0$,
3. defined modulo any function vanishing on the null cone, $\Phi(X) \sim \Phi(X) + V^2(X)\alpha(X)$,

Conformal scalar field in higher-spin background

The Weyl symbol $H(X,P)$ of the ambient Hermitian differential operator $\hat{H}$ can be interpreted as a function over the ambient phase space $\Gamma(\odot T\mathcal{M}_{d+2}) \subset C^\infty(T^*\mathcal{M}_{d+2})$ whose expansion in powers of the momenta are ambient lift of (conformal) higher-spin gauge fields on the (conformal boundary of the) spacetime manifold.

Isomorphism of spectra

AdS/CFT dictionary: intertwiner

- of the unitary irreductible modules of observables (state/operators)
- of the Lie algebra $\mathfrak{so}(d,2)$ of symmetries (isometry/conformal)

Conformal scalar field in higher-spin background

More generally, one can consider 3 differential operators spanning the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ which are deformations of the lower-order ones.

The linearisation of this system can be shown to describe an infinite tower of off-shell (conformal) higher-spin gauge fields on $EAdS_{d+1}$ (on S^d).

Imposing these ambient fields to be harmonic leads to Frønsdal equations on $EAdS_{d+1}$ (and Fradkin-Tseytlin equations on S^d).

Klein geometry

Definition: *A closed Lie subgroup $H \subseteq G$ of a connected Lie group G is called a Klein geometry with principal group G, isotropy group H and homogeneous space G/H.*

Reminder:

- The principal group G can be seen as a principal H-bundle over the homogeneous space G/H.
- The tangent bundle of G is trivial: $TG \cong G \times \mathfrak{g}$.

Klein geometry

Definition: *A closed Lie subgroup $H \subseteq G$ of a connected Lie group G is called a Klein geometry with principal group G, isotropy group H and homogeneous space G/H.*

Reminder:

Let us denote by $\omega^{\mathfrak{g}} : TG \to \mathfrak{g}$ the Maurer-Cartan one-form on G.

1. It is equivariant.
2. It defines an absolute parallelism $\omega^{\mathfrak{g}}|_g : T_gG \to \mathfrak{g}$, i.e. a moving frame $g \mapsto \omega^{\mathfrak{g}}|_g$ on G.
   Equivalently, it defines the diffeomorphism $TG \to G \times \mathfrak{g}$.

Main idea: Cartan geometries are "curved" versions of Klein geometries.

Cartan geometry

Definition: *Let $H \subseteq G$ be a Klein geometry. Let P be a principal H-bundle over $M \cong P/H$.*
A Cartan connection $\omega^{\mathfrak{g}} : TG \to \mathfrak{g}$ is a $\mathfrak{g}$-valued one-form on P such that

1. *It is equivariant.*
2. *It defines an absolute parallelism $\omega^{\mathfrak{g}}|_p : T_pP \to \mathfrak{g}$, i.e. a moving frame $p \mapsto \omega^{\mathfrak{g}}|_p$ on P.*
3. *Its restriction to the vertical sub-bundle $VP \subset TP$ reduces to the canonical isomorphism between the quotient VP/H and the adjoint bundle $P \times_H \mathfrak{h}$ over P/H.*

Cartan geometry

A Cartan connection always defines a solder form $\omega^{\mathfrak{g}/\mathfrak{h}} = \omega^{\mathfrak{g}}\ mod\ \mathfrak{h}$, hence a bundle homomorphism from TM to $P \times_H \mathfrak{g}/\mathfrak{h}$.

Cartan geometry

Given a Cartan geometry (i.e. a Klein geometry, a principal bundle and a Cartan connection), let σ be a local section of the principal H-bundle P.

Then the pullback $\sigma^*\omega^{\mathfrak{g}}$ of the Cartan connection is a $\mathfrak{g}$-valued one-form on a chart of the base manifold M (hence the pullback of the solder form $\sigma^*\omega^{\mathfrak{g}/\mathfrak{h}}$ defines a coframe field).

Cartan geometry

Definition: *A Klein geometry $H \subset G$ is reductive if $\mathfrak{h}$ has an H-invariant complement $\mathfrak{p}$ ($\cong \mathfrak{g}/\mathfrak{h}$).*

The restriction $\omega^{\mathfrak{h}}$ of the codomain of a Cartan connection one-form $\omega^{\mathfrak{g}}$ to the isotropy subalgebra $\mathfrak{h}$ of a reductive Klein geometry is an Ehresmann connection on the principal H-bundle P.

Examples of Klein geometries

Klein geometry	Isotropy group H	Principal group G	Homogeneous space G/H
Affine	$GL(n\mathbb{R})$	$IGL(n\mathbb{R})$	$\mathbb{R}^n$
Euclidean	$SO(n)$	$ISO(n)$	$\mathbb{R}^n$
Möbius	$O(n+1,1)$	Parabolic	S^n

Examples of Cartan geometries

Cartan geometry	Klein model	Cartan connection
Differential	Affine	Affine
Riemannian	Euclidean	Levi-Civita
Conformal	Möbius	Cartan

Cartan-like geometry

Recipe: Replace the isometry group $\mathfrak{so}(d,2)$ of AdS_{d+1} in the Cartan connection one-form by the infinite-dimensional Lie algebra $\mathfrak{hs}(d,2)$ containing $\mathfrak{so}(d,2)$ as a subalgebra. Try to construct actions actions or nonlinear equations, written in terms of the curvature two-form, such that their linearisation reproduces the free theory.

Some simple interacting theories (too simple?)

The actions of the following topological gravity theories:

- Chern-Simons ($D = 3$),
- Conformal ($D = 3$),
- Jackiw-Teitelboim ($D = 2$),

admit a formulation in terms of Cartan connections with values in one (or two) copy (copies) of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Upon replacement with the special linear algebras $\mathfrak{sl}(N, \mathbb{R})$ or their infinite-dimensional counterparts $\mathfrak{hs}[\lambda]$, one obtains fully interacting higher-spin gravity theories without matter.

Some simple interacting theories (too simple?)

There exists at least 4 examples of fully interacting higher-spin gravity theories without matter extending the following spin-two gravity theories:

- Chern-Simons ($D = 3$),
- Conformal (even D or Chern-Simons for $D = 3$),
- Jackiw-Teitelboim ($D = 2$),
- Self-Dual ($D = 4$).

They share the following features: they

- are consistent with minimal coupling to gravity even around flat background,
- do not require the unfolding procedure,
- can be truncated consistently to the usual low-spin sector,
- admit a perturbatively local action principle.

But they are either topological, not unitary or in Euclidean signature.

Jet prolongation and unfolding

Higher-spin gravity theories with unitary propagating degrees of freedom require much more efforts and are only known at the level of equations in "unfolded" form, i.e. as L_∞-algebras (Lie algebroids).

Conclusion

Higher-spin gauge fields have a long history. Understanding better the properties of their interactions remains a well-motivated major challenge of theoretical physics, touching subjects at the forefront of contemporary mathematical physics (such as conformal geometry, deformation quantisation, geometry of PDEs, representation theory, $\mathbb{Z}$-graded manifolds, topological field theory, etc).