$L_\infty$-Algebras of Gravity
and their Braided Deformations

Richard Szabo
Outline

▶ Introduction/Motivation

▶ $L_\infty$-algebras and classical field theories

▶ Einstein-Cartan-Palatini (ECP) gravity and its $L_\infty$-algebras

▶ Noncommutative ECP gravity and braided $L_\infty$-algebras

with M. Dimitrijević Ćirić, G. Giotopoulos & V. Radovanović
Nonassociative Gravity?

- In certain non-geometric flux compactifications of string theory, low-energy effective dynamics of closed strings may be described by noncommutative or even nonassociative deformations of gravity

  (Blumenhagen & Plauschinn '10; Lüst '10; Mylonas, Schupp & Sz '12; ...)

- Metric aspects of nonassociative differential geometry only partially developed, no version of the Einstein-Hilbert action is known

  (Blumenhagen & Fuchs '16; Aschieri, Dimitrijević Ćirić & Sz '17)

- Try to treat as a deformation of 'gauge theory':
  
  Use Einstein-Cartan principal bundle formulation, corresponding action is the Palatini action

  (Barnes, Schenkel & Sz '16)

- Problems with naive definition of gauge transformations:

  $\delta \star \alpha A = d \alpha + [\alpha, A] \star = d \alpha + \alpha \star A - A \star \alpha$

  Nonassociativity obstructs closure of gauge algebra:

  $(\delta \star \alpha \delta \star \beta - \delta \star \beta \delta \star \alpha) A \neq \delta \star [\alpha, \beta] \star A$
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  \[ (\delta^*_{\alpha} \delta^*_{\beta} - \delta^*_{\beta} \delta^*_{\alpha})A \neq \delta^*_{[\alpha, \beta]} A \]
Higher spin gauge theories with field-dependent gauge parameters:

\[ (\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha) \Phi = \delta_{C(\alpha,\beta,\Phi)} \Phi \]

(Berends, Burgers & van Dam ’85)

"Generalized" gauge symmetries of closed string field theory involve higher brackets:

\[ \delta_\alpha \Phi = \sum_{n, \ell} n_{\ell}(\alpha, \Phi)_{n-1} \]

(Zwiebach ’92)

Dual to differential graded (commutative) algebras

(Lada & Stasheff ’92)

Deformation theory: Kontsevich’s Formality Theorem based on \( L_\infty \)-quasi-isomorphisms of differential graded Lie algebras

Any classical field theory with "generalized" gauge symmetries is determined by an \( L_\infty \)-algebra, due to duality with BV–BRST

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$L_\infty$-Algebras in Physics & Mathematics

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$L_\infty$-Algebras: Gravity/Noncommutative Gauge Theory

- $L_\infty$-algebras of Einstein-Hilbert gravity: Requires perturbation about flat background, involves infinitely-many brackets
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- $L_\infty$-algebras of noncomm./nonass. gauge theories typically also require infinitely-many brackets
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- **In this talk:** Explain $L_\infty$-algebra formulation of ECP gravity, define deformation with braided gauge symmetries, and then present braided $L_\infty$-algebra determining noncommutative gravity
What is an $L_\infty$-Algebra?

- Graded vector space: $V = \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots$, with graded exterior algebra $\Lambda_V = \wedge^\bullet(V[1])$ viewed as a free cocommutative coalgebra

- $L : \Lambda_V \longrightarrow \Lambda_V$ coderivation of degree $|L| = 1$, with $L^2 = 0$
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- Write $L^2 = 0$ in ‘components’ $L = \{\ell_n\}$ where $\ell_n : \wedge^n(V[1]) \longrightarrow V[1]$ with $|\ell_n| = 1$, or restoring original grading $\ell_n : \wedge^n V \longrightarrow V$ with $|\ell_n| = 2 - n$:

  $\ell_1(\ell_1(v)) = 0$  
  $(V, \ell_1)$ is a cochain complex

  $\ell_1(\ell_2(v, w)) = \ell_2(\ell_1(v), w) \pm \ell_2(v, \ell_1(w))$  
  $\ell_1$ is a derivation of $\ell_2$

  $\ell_2(v, \ell_2(w, u)) + \text{cyclic} = (\ell_1 \circ \ell_3 \pm \ell_3 \circ \ell_1)(v, w, u)$  
  Jacobi up to homotopy

  plus “higher homotopy Jacobi identities”

Dualizing gives graded commutative algebra derivation $Q = L^* : \Lambda^* V \longrightarrow \Lambda^* V$ with $|Q| = 1$, $Q^2 = 0$. $L_\infty$-algebras are generalizations of differential graded Lie algebras.
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$L_\infty$-Quasi-Isomorphisms

- **$L_\infty$-morphism**: Degree-preserving coalgebra homomorphism
  \[ \Psi : \Lambda_V \rightarrow \Lambda_{V'} \]
  intertwining codifferentials:
  \[ \Psi \circ L = L' \circ \Psi \]
\(L_\infty\)-Quasi-Isomorphisms

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- In ‘components’ \(\Psi = \{\psi_n\}\) where \(\psi_n : \wedge^n V \longrightarrow V'\) with \(|\psi_n| = 1 - n\):
  \[\psi_1 \ell_1 = \ell'_1 \psi_1\]
  \(\psi_1\) is a cochain map
  \[\psi_1(\ell_2(v, w)) - \ell'_2(\psi_1(v), \psi_1(w)) = \text{homotopy in } \psi_2\]
  plus cumbersome higher relations
\textbf{$L_\infty$-Quasi-Isomorphisms}

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- Quasi-isomorphism is an equivalence relation on \( L_\infty \)-algebras
  (contrary to dg Lie algebras)
Cyclic $L_\infty$-Algebras

- Cyclic pairing $\langle - , - \rangle : V \times V \rightarrow \mathbb{R}$ is non-degenerate, graded symmetric, bilinear and satisfies cyclicity:

$$\langle v_0, \ell_n(v_1, v_2, \ldots, v_n) \rangle = \pm \langle v_1, \ell_n(v_0, v_2, \ldots, v_n) \rangle$$
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- Cyclic $L_\infty$-morphisms $\Psi : \Lambda_V \to \Lambda_{V'}$ preserve cyclic pairings:

  $$\langle \psi_1(v), \psi_1(w) \rangle' = \langle v, w \rangle$$

  $$\sum_{i=1}^{n-1} \langle \psi_i(v_1, \ldots, v_i), \psi_{n-i}(v_{i+1}, \ldots, v_n) \rangle' = 0$$
$L_\infty$-Algebras of Classical Field Theories

- BV formalism constructs a dg algebra $(\mathcal{C}_\infty(V[1]), Q_{BV})$ on graded vector space $V$ of BV fields (ghosts, fields and antifields)
**L∞-Algebras of Classical Field Theories**

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- Translate coordinate functions \(\xi\) to elements of vector spaces, then action of \(Q_{BV}\) is a polynomial in ghosts, fields and antifields and their derivatives, dual to sum over all brackets \(\ell_n\) on \(V\):

\[
Q_{BV}\xi = \ell_1(\xi) + \frac{1}{2} \ell_2(\xi, \xi) + \cdots
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- BV symplectic form (inducing antibracket) of degree $-1$ on $V$ induces cyclic pairing of degree $-3$

  \[
  \cdots V_0 \quad V_1 \quad V_2 \quad V_3 \quad \cdots
  \]  
  \[
  \cdots \text{gauge par. fields field eqs. Noether ids. } \cdots
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- $V_{-k}$ encode ‘higher gauge transformations’ (ghosts-for-ghosts, etc.) for reducible symmetries
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- **Gauge transformations** of fields $A \in V_1$ by $\lambda \in V_0$:
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- **Closure of gauge algebra**:
  \[
  [\delta_\lambda_1, \delta_\lambda_2]A = \delta_{C(\lambda_1, \lambda_2; A)} A
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- \textbf{Field equations}:
  \[
  \mathcal{F}_A = \ell_1(A) - \frac{1}{2} \ell_2(A, A) + \cdots
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- **Field equations**: \[ F_A = \ell_1(A) - \frac{1}{2} \ell_2(A, A) + \cdots \]

- **Noether identities**: \[ I_\lambda = \ell_1(F_A) + \ell_2(F_A, A) + \cdots = 0 \text{ (off-shell)} \]
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  \[ I_\lambda = \ell_1(F_A) + \ell_2(F_A, A) + \cdots = 0 \text{ (off-shell)} \]

- **Action**:
  \[ S = \frac{1}{2} \langle A, \ell_1(A) \rangle - \frac{1}{3!} \langle A, \ell_2(A, A) \rangle + \cdots \]
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  \mathcal{I}_\lambda = \ell_1(\mathcal{F}_A) + \ell_2(\mathcal{F}_A, A) + \cdots = 0 \text{ (off-shell)}
  \]

- Action:
  \[
  S = \frac{1}{2} \langle A, \ell_1(A) \rangle - \frac{1}{3!} \langle A, \ell_2(A, A) \rangle + \cdots
  \]

- Moduli space = field equations / gauge transformations
$L_\infty$-Algebras of Classical Field Theories

- **Gauge transformations** of fields $A \in V_1$ by $\lambda \in V_0$:
  $$\delta_\lambda A = \ell_1(\lambda) + \ell_2(\lambda, A) + \cdots$$

- **Closure of gauge algebra**:
  $$[\delta_{\lambda_1}, \delta_{\lambda_2}]A = \delta_{\mathcal{C}(\lambda_1, \lambda_2; A)}A$$
  $$\mathcal{C}(\lambda_1, \lambda_2; A) = \ell_2(\lambda_1, \lambda_2) + \ell_3(\lambda_1, \lambda_2, A) + \cdots$$

- **Field equations**:
  $$\mathcal{F}_A = \ell_1(A) - \frac{1}{2} \ell_2(A, A) + \cdots$$

- **Noether identities**:
  $$\mathcal{I}_\lambda = \ell_1(\mathcal{F}_A) + \ell_2(\mathcal{F}_A, A) + \cdots = 0 \text{ (off-shell)}$$

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- **Moduli space** = field equations / gauge transformations

- **Quasi-isomorphic $L_\infty$-algebras give equivalent field theories**
Example: Chern-Simons Theory

- $\dim(M) = 3$, $\mathfrak{g} =$ quadratic Lie algebra with pairing $\text{Tr}_g$
Example: Chern-Simons Theory

- $\dim(M) = 3$, $\mathfrak{g}$ = quadratic Lie algebra with pairing $\text{Tr}_\mathfrak{g}$
- Cochain complex = (de Rham complex) $\otimes \mathfrak{g}$: $V = \Omega^\bullet(M, \mathfrak{g})$
- Brackets: $\ell_1 = d$, $\ell_2 = [-,-]_\mathfrak{g}$
- Cyclic pairing: $\langle \alpha, \beta \rangle = \int_M \text{Tr}_\mathfrak{g}(\alpha \wedge \beta)$
Example: Chern-Simons Theory

- \( \dim(M) = 3 \), \( g \) = quadratic Lie algebra with pairing \( \text{Tr}_g \)
- Cochain complex = (de Rham complex) \( \otimes g \): \( V = \Omega^\bullet(M, g) \)
- Brackets: \( \ell_1 = d \), \( \ell_2 = [-,-]_g \)
- Cyclic pairing: \( \langle \alpha, \beta \rangle = \int_M \text{Tr}_g(\alpha \wedge \beta) \)
- Field equations for \( A \in V_1 = \Omega^1(M, g) \): \( \mathcal{F}_A = dA + \frac{1}{2} [A, A]_g \)
- Moduli space = flat connections on \( M \)
- Noether identity = Bianchi identity: \( \mathcal{I}_\lambda = d\mathcal{F}_A + [\mathcal{F}_A, A]_g = 0 \)
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- Chern-Simons gauge theory is organised by a dg Lie algebra
**Einstein-Cartan-Palatini Gravity (4d)**

\[ S = \int_M \text{Tr}(e \wedge e \wedge R) = \int_M \varepsilon_{abcd} (e^a \wedge e^b \wedge R^{cd}) \]  

**Fields:**  
\( e : TM \rightarrow V \) bundle isomorphism onto ‘fake tangent bundle’ \( V \) with Minkowski metric \( \eta \), defines coframe \( e \in \Omega^1(M, V) \)  
\( R = d\omega + \frac{1}{2} [\omega, \omega] \in \Omega^2(M, P \times_{\text{ad}} \mathfrak{so}(1, 3)) \) curvature of spin connection \( \omega \) on associated principal \( SO(1, 3) \)-bundle \( P \rightarrow M \)  
\( \text{Tr} : \Omega^4(M, \wedge^4 V) \rightarrow \Omega^4(M) \)
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\[ \text{Tr} : \Omega^4(M, \wedge^4 \mathcal{V}) \rightarrow \Omega^4(M) \]

- **Locally, or globally if \( M \) parallelizable:**

\[ e \in \Omega^1(M, \mathbb{R}^{1,3}) , \quad \omega \in \Omega^1(M, \mathfrak{so}(1, 3)) , \quad \text{Tr} : \wedge^4(\mathbb{R}^{1,3}) \rightarrow \mathbb{R} \]
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\[
R = d\omega + \frac{1}{2} [\omega, \omega] \in \Omega^2(M, P \times_{\text{ad}} \mathfrak{so}(1, 3)) \quad \text{curvature of spin connection} \quad \omega \quad \text{on associated principal} \quad SO(1, 3)-\text{bundle} \quad P \longrightarrow M
\]

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▶ Locally, or globally if \( M \) parallelizable:

\( e \in \Omega^1(M, \mathbb{R}^{1,3}) \), \( \omega \in \Omega^1(M, \mathfrak{so}(1, 3)) \), \( \text{Tr} : \wedge^4(\mathbb{R}^{1,3}) \longrightarrow \mathbb{R} \)

▶ Bianchi identities: \( d\omega \, T = R \wedge e \), \( d\omega \, R = 0 \)

\[
T = d\omega \, e = de + \omega \wedge e = \text{torsion of} \, \omega
\]
Einstein-Cartan-Palatini Gravity (4d)

- (Infinitesimal) gauge symmetries: Diffeos + local Lorentz
  \[ \Gamma(TM) \rtimes \Omega^0(M, \mathfrak{so}(1, 3)) \]
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For \( e \) non-degenerate, equivalent to torsion-free + vacuum Einstein equations
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- In any dimension \( d \):
  \[ e^2 \rightarrow e^{d-2} \text{ in action} \]

- For \( d = 3 \):
  \[ T = R = 0 \]

Note: In contrast to Einstein-Hilbert formulation, ECP theory makes sense for degenerate coframes (required for \( \mathbb{L}_\infty \)-algebra formulation).
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- **Note:** In contrast to Einstein-Hilbert formulation, ECP theory makes sense for degenerate coframes \( e \) (required for \( L_\infty \)-algebra formulation)
\textit{L}_\infty\text{-Algebra Picture of ECP Gravity (3d)}

- **Cochain complex:** \( V_0 \xrightarrow{\ell_1} V_1 \xrightarrow{\ell_1} V_2 \xrightarrow{\ell_1} V_3 \)

- **Gauge transformations:** \((\xi, \lambda) \in V_0 = \Gamma(TM) \times \Omega^0(M, so(3))\)

- **Physical fields:** \((e, \omega) \in V_1 = \Omega^1(M, \mathbb{R}^3) \times \Omega^1(M, so(3))\)

- **Field equations:** \((E, \Omega) \in V_2 = \Omega^2(M, \Lambda^2(\mathbb{R}^3)) \times \Omega^2(M, \mathbb{R}^3)\)

- **Noether identities:** \((\Xi, \Lambda) \in V_3 = \Omega^1(M, \Omega^3(M)) \times \Omega^3(M, \mathbb{R}^3)\)

- **Differential:** \(\ell_1(\xi, \lambda) = (0, d\lambda)\) \(\ell_1(e, \omega) = (0, 0)\) \(\ell_1(E, \Omega) = (0, d\Omega)\)
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- **Higher brackets:**

\[
\ell_2((\xi_1, \lambda_1), (\xi_2, \lambda_2)) = ([\xi_1, \xi_2], -[\lambda_1, \lambda_2] + \mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1)
\]

\[
\ell_2((\xi, \lambda), (e, \omega)) = (-\lambda \cdot e + \mathcal{L}_\xi e, -[\lambda, \omega] + \mathcal{L}_\xi \omega)
\]

\[
\ell_2((e_1, \omega_1), (e_2, \omega_2)) = -(2 \omega_2 \wedge \omega_1, \omega_1 \wedge e_2 + \omega_2 \wedge e_1)
\]

+ more \(\ell_2\) related to field equations and Noether identities
$L_\infty$-Algebra Picture of ECP Gravity (3d)

- 3d gravity is organised by a dg Lie algebra:
  - Gauge symmetry:
    \[
    \delta_{(\xi,\lambda)}(e,\omega) = (-\lambda \cdot e + \mathcal{L}_\xi e, \ d\lambda - [\lambda, \omega] + \mathcal{L}_\xi \omega) = \ell_1(\xi, \lambda) + \ell_2((\xi, \lambda), (e, \omega))
    \]
  - Field equations: $\mathcal{F}_{(e,\omega)} = (R, T) = \ell_1(e, \omega) - \frac{1}{2} \ell_2((e, \omega), (e, \omega))$
  - Noether identities:
    \[
    \mathcal{I}_{(\xi,\lambda)} = \left( dx^\mu \otimes \text{Tr}(\nu_{\partial_\mu} e \wedge dR - \nu_{\partial_\mu} de \wedge R) + (e \leftrightarrow \omega), \ d\omega \ T - R \wedge e \right) \\
    = \ell_1(\mathcal{F}_{(e,\omega)}) - \ell_2((e, \omega), \mathcal{F}_{(e,\omega)}) = (0, 0) \ (\text{off-shell})
    \]
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  \]

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  = \ell_1(\mathcal{F}(e, \omega)) - \ell_2((e, \omega), \mathcal{F}(e, \omega)) = (0, 0) \quad \text{(off-shell)}
  \]

- **Cyclic pairing:**
  \[
  \langle (e, \omega), (E, \Omega) \rangle := \int_M \text{Tr}(e \wedge E + \Omega \wedge \omega) \quad \text{encodes:}
  \]
  - **ECP action:**
    \[
    S = \langle (e, \omega), \ell_1(e, \omega) + \ell_2((e, \omega), (e, \omega)) \rangle
    \]
  - **Extend to** \( \langle - , - \rangle : \mathcal{V}_0 \times \mathcal{V}_3 \rightarrow \mathbb{R} \) using gauge invariance and integration by parts; then cyclicity on \( \mathcal{V}_0 \) implies Noether identities.
\textit{\(L_\infty\)-Algebra Picture of ECP Gravity (3d)}

- 3d gravity is organised by a dg Lie algebra:
  \[ \delta_{(\xi,\lambda)}(e, \omega) = (-\lambda \cdot e + \mathcal{L}_\xi e, \; d\lambda - [\lambda, \omega] + \mathcal{L}_\xi \omega) = \ell_1(\xi, \lambda) + \ell_2((\xi, \lambda), (e, \omega)) \]

- Gauge symmetry:
  \[ F_{(e, \omega)} = (R, T) = \ell_1(e, \omega) - \frac{1}{2} \ell_2((e, \omega), (e, \omega)) \]

- Field equations: \( I_{(\xi, \lambda)} = (dx^\mu \otimes \text{Tr}(\iota_{\partial_\mu} e \wedge dR - \iota_{\partial_\mu} de \wedge R) + (e \leftrightarrow \omega), \; d\omega T - R \wedge e) = \ell_1(F_{(e, \omega)}) - \ell_2((e, \omega), F_{(e, \omega)}) = (0, 0) \quad \text{(off-shell)} \]

- Noether identities:

- Cyclic pairing: \( \langle (e, \omega), (E, \Omega) \rangle := \int_M \text{Tr}(e \wedge E + \Omega \wedge \omega) \) encodes:
  - ECP action: \( S = \langle (e, \omega), \ell_1(e, \omega) + \ell_2((e, \omega), (e, \omega)) \rangle \)

- Extend to \( \langle - , - \rangle : V_0 \times V_3 \rightarrow \mathbb{R} \) using gauge invariance and integration by parts; then cyclicity on \( V_0 \) implies Noether identities

- Describes BV–BRST formulation of ECP gravity
  (with higher brackets for \( d \geq 4 \))

(Cattaneo & Schiavina '17)
Application: Chern-Simons Gravity

- 3d gravity is equivalent to Chern-Simons theory with gauge algebra
  \[ g = \text{iso}(3) = \mathbb{R}^3 \ltimes \mathfrak{so}(3), \quad A = (e, \omega), \quad \mathcal{F}_A = (R, T) \]  
  (Witten '88)
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- For invertible \( e \), diffeos \( \xi \in \Gamma(TM) \) are equivalent on-shell to gauge transfs by \( (\tau_\xi, \lambda_\xi) = (\iota_\xi e, \iota_\xi \omega) \in \Omega^0(M, g) \):
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  $$\delta_\xi A = L_\xi A = \delta(\tau_\xi, \lambda_\xi)A + \iota_\xi \mathcal{F}_A$$

- Diffeos are redundant symmetries: trivialise by extending $V_0 (V_3)$ by extra “shift” symmetries $\Omega^0(M, \mathbb{R}^3)$ ($\Omega^3(M, \mathfrak{so}(3))$), and adding $V_{-1} = \Gamma(TM) (V_4 = \Omega^1(M, \Omega^3(M))$ with $\ell_1 : V_{-1} \hookrightarrow V_0 (\ell_1 : V_3 \twoheadrightarrow V_4)$; then $H^\bullet(V_{\text{Ext}}^{\text{ECP}}, \ell_1) \cong H^\bullet(V_{\text{CS}}, \ell_1)$
Application: Chern-Simons Gravity

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- There is an (off-shell) cyclic \( L_\infty \)-quasi-isomorphism \( \{ \psi_n \} \) with \( \psi_n = 0 \) for \( n \geq 3 \) from the Chern-Simons dg Lie algebra to the ECP dg Lie algebra
Braided Noncommutative Deformation

Let \( \mathcal{F} = f^\alpha \otimes f_\alpha \in U\Gamma(TM) \otimes U\Gamma(TM) \) be a Drinfel’d twist;

- e.g. Moyal-Weyl twist \( \mathcal{F} = \exp\left(-\frac{i}{2} \theta^{\mu \nu} \partial_\mu \otimes \partial_\nu \right) \)
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If $A$ is a $U\Gamma(TM)$-module algebra, deform product on $A$ into a star-product:

$$a \star b = \mathcal{F}^{-1}(a \otimes b) = \tilde{f}^\alpha(a) \cdot \tilde{f}_\alpha(b)$$
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If \( \mathcal{A} \) is a \( U\Gamma(TM) \)-module algebra, deform product on \( \mathcal{A} \) into a star-product:

\[
a \star b = \mathcal{F}^{-1}(a \otimes b) = \bar{f}^\alpha(a) \cdot \bar{f}_\alpha(b)
\]

Defines noncommutative algebra \( \mathcal{A}_\star \) carrying representation of twisted Hopf algebra \( U_{\mathcal{F}} \Gamma(TM) \):

\[
\xi(a \star b) = \xi_{(1)}(a) \star \xi_{(2)}(b) , \quad \Delta(\xi) = \xi_{(1)} \otimes \xi_{(2)}
\]
Braided Noncommutative Deformation

Let \( \mathcal{F} = f^{\alpha} \otimes f_{\alpha} \in \mathcal{U} \Gamma(TM) \otimes \mathcal{U} \Gamma(TM) \) be a Drinfel’d twist; e.g. Moyal-Weyl twist
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\xi(a \star b) = \xi_{(1)}(a) \star \xi_{(2)}(b) \ , \ \Delta(\xi) = \xi_{(1)} \otimes \xi_{(2)}
\]

If \( \mathcal{A} \) is commutative, then \( \mathcal{A}_{\star} \) is braided-commutative:
\[
a \star b = \bar{R}^{\alpha}(b) \star \bar{R}_{\alpha}(a)
\]
\[
\mathcal{R} = \mathcal{F}^{-2} = R^{\alpha} \otimes R_{\alpha} = \text{triangular } \mathcal{R}-\text{matrix}
\]
Braided Noncommutative Deformation

- Let $\mathcal{F} = f^\alpha \otimes f^\alpha \in U\Gamma(TM) \otimes U\Gamma(TM)$ be a Drinfel’d twist; e.g. Moyal-Weyl twist $\mathcal{F} = \exp\left(-\frac{i}{2} \theta^{\mu\nu} \partial_\mu \otimes \partial_\nu\right)$

- If $\mathcal{A}$ is a $U\Gamma(TM)$-module algebra, deform product on $\mathcal{A}$ into a star-product:
  \[ a \star b = \cdot \mathcal{F}^{-1}(a \otimes b) = \bar{f}_\alpha(a) \cdot \bar{f}_\alpha(b) \]

- Defines noncommutative algebra $\mathcal{A}_\star$ carrying representation of twisted Hopf algebra $U\mathcal{F}\Gamma(TM)$:
  \[ \xi(a \star b) = \xi(1)(a) \star \xi(2)(b) \quad , \quad \Delta(\xi) = \xi(1) \otimes \xi(2) \]

- If $\mathcal{A}$ is commutative, then $\mathcal{A}_\star$ is braided-commutative:
  \[ a \star b = \bar{R}_\alpha(b) \star \bar{R}_\alpha(a) \]
  \[ \mathcal{R} = \mathcal{F}^{-2} = R^\alpha \otimes R_\alpha = \text{triangular } \mathcal{R}-\text{matrix} \]

- More generally, if $\mathcal{F}$ is a cochain twist, then $U\mathcal{F}\Gamma(TM)$ is a quasi-Hopf algebra and $\mathcal{A}_\star$ is a nonassociative algebra
Braided Gauge Symmetry

- Braided Lie algebra $\Omega^0_\star(M, so(3))$: $[\lambda_1, \lambda_2]_\star := [-, -] \circ F^{-1}(\lambda_1 \otimes \lambda_2)$
- Braided antisymmetric: $[\lambda_1, \lambda_2]_\star = -[\bar{R}^\alpha \lambda_2, \bar{R}_\alpha \lambda_1]_\star$, braided Jacobi
Braided Gauge Symmetry

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- **Braided spin connection, coframe** \( \omega \in \Omega^1_*(M, \mathfrak{so}(3)) \), \( e \in \Omega^1_*(M, \mathbb{R}^3) \) transform in braided representations:
  \[ \delta_\lambda^* e = -\lambda \ast e \quad , \quad \delta_\lambda^* \omega = d\lambda - [\lambda, \omega]_* \]
Braided Gauge Symmetry

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- **Braided antisymmetric**: $[\lambda_1, \lambda_2]_* = -[\bar{R}^\alpha \lambda_2, \bar{R}_\alpha \lambda_1]_*$, braided Jacobi
- **Braided spin connection, coframe** $\omega \in \Omega^1_*(M, \mathfrak{so}(3))$, $e \in \Omega^1_*(M, \mathbb{R}^3)$ transform in braided representations:
  \[ \delta^*_\lambda e = -\lambda \star e \quad , \quad \delta^*_\lambda \omega = d\lambda - [\lambda, \omega]_* \]
- **Braided gauge transformations** satisfy braided Leibniz rule:
  \[ \delta^*_\lambda (e \otimes \omega) = \delta^*_\lambda e \otimes \omega + \bar{R}^\alpha e \otimes \delta^*_{\bar{R}_\alpha} \lambda \omega \]
Braided Gauge Symmetry

- **Braided Lie algebra** $\Omega^0(M, \mathfrak{so}(3))$: $[\lambda_1, \lambda_2]_\star := [-, -] \circ \mathcal{F}^{-1}(\lambda_1 \otimes \lambda_2)$
- **Braided antisymmetric**: $[\lambda_1, \lambda_2]_\star = -[\overline{R}^\alpha \lambda_2, \overline{R}_\alpha \lambda_1]_\star$, braided Jacobi
- **Braided spin connection, coframe** $\omega \in \Omega^1_\star(M, \mathfrak{so}(3))$, $e \in \Omega^1_\star(M, \mathbb{R}^3)$ transform in braided representations:
  $$\delta_\lambda^\star e = -\lambda \star e, \quad \delta_\lambda^\star \omega = d\lambda - [\lambda, \omega]_\star$$
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- **Braided covariant derivative** gives braided curvature and torsion:
  $$R := d\omega + \frac{1}{2} [\omega, \omega]_\star, \quad T := de + \omega \wedge_\star e$$
Braided Gauge Symmetry

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Braided ECP Gravity

\[ S^* = \int_M \text{Tr}(e \wedge_\star R) = \int_M \varepsilon_{abc} (e^a \wedge_\star R^{bc}) \]
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  \[ \delta^*_\lambda R[\omega] \neq R[\omega + \delta^*_\lambda \omega] \]
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- Field equations are braided covariant, but braided gauge symmetries do not produce new solutions:
  \( \delta^*_\chi R[\omega] \neq R[\omega + \delta^*_\chi \omega] \)

- Braided version of Noether’s Second Theorem gives “braided” Noether identities *off-shell*, justifies interpretation of local braided symmetries as “gauge”
Braided $L_\infty$-Algebras

If $(V, \{\ell_n\})$ is a classical $L_\infty$-algebra in the category of $U\Gamma(TM)$-modules, then $(V, \{\ell^*_n\})$ is a braided $L_\infty$-algebra in the category of $U\mathcal{F}\Gamma(TM)$-modules, where

$$\ell^*_n(v_1 \wedge \cdots \wedge v_n) := \ell_n(v_1 \wedge^* \cdots \wedge^* v_n)$$
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Braided graded antisymmetry:

$$\ell^*_n(\ldots, v, v', \ldots) = -(-1)^{|v||v'|} \ell_n^*(\ldots, \bar{R}^\alpha(v'), \bar{R}_\alpha(v), \ldots)$$

+ braided homotopy Jacobi identities (unchanged for $n = 1, 2$)
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+ braided homotopy Jacobi identities (unchanged for $n = 1, 2$)

- Cyclic pairing: $\langle -,- \rangle_\ast := \langle -,- \rangle \circ \mathcal{F}^{-1}$
Braided $L_\infty$-Algebra of Noncommutative Gravity

For braided ECP gravity, underlying cochain complex $(V, \ell_1)$ is formally unchanged.
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- For braided ECP gravity, underlying cochain complex $(V, \ell_1)$ is formally unchanged

- Higher brackets:

  \[
  \ell^*_2((\xi_1, \lambda_1), (\xi_2, \lambda_2)) = ([\xi_1, \xi_2]_*, -[\lambda_1, \lambda_2]_* + \mathcal{L}^*_\xi_1 \lambda_2 - \mathcal{L}^*_{\bar{R}_\alpha \xi_2} \bar{R}_\alpha \lambda_1)
  \]

  \[
  \ell^*_2((\xi, \lambda), (e, \omega)) = (-\lambda \ast e + \mathcal{L}^*_\xi e, -[\lambda, \omega]_* + \mathcal{L}^*_\xi \omega)
  \]

  \[
  \ell^*_2((e_1, \omega_1), (e_2, \omega_2)) = -(2 \omega_2 \wedge_* \omega_1, \omega_1 \wedge_* e_2 + \bar{R}^\alpha \omega_2 \wedge_* \bar{R}_\alpha e_1)
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  + more $\ell^*_2$ related to field equations and Noether identities
Braided $L_\infty$-Algebra of Noncommutative Gravity

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- Higher brackets:
  \[
  \begin{align*}
  \ell_2^*((\xi_1, \lambda_1), (\xi_2, \lambda_2)) &= ([\xi_1, \xi_2]_*, -[\lambda_1, \lambda_2]_* + \mathcal{L}_{\xi_1}^* \lambda_2 - \mathcal{L}_{\bar{R}_\alpha \xi_2}^* \bar{R}_\alpha \lambda_1) \\
  \ell_2^*((\xi, \lambda), (e, \omega)) &= (-\lambda \star e + \mathcal{L}_{\xi}^* e, -[\lambda, \omega]_* + \mathcal{L}_{\xi}^* \omega) \\
  \ell_2^*((e_1, \omega_1), (e_2, \omega_2)) &= -(2 \omega_2 \wedge_* \omega_1, \omega_1 \wedge_* e_2 + \bar{R}_\alpha \omega_2 \wedge_* \bar{R}_\alpha e_1)
  \end{align*}
  \]
  + more $\ell_2^*$ related to field equations and Noether identities.

- Organises all dynamics of 3d noncommutative gravity:
  - Gauge symmetry and field equations as classically $(A = (e, \omega))$:
    \[
    \delta_{(\xi, \lambda)}^* A = \ell_1^*(A) + \ell_2^*((\xi, \lambda), A) , \quad \mathcal{F}_A^* = \ell_1^*(A) - \frac{1}{2} \ell_2^*(A, A)
    \]
  - Noether identities due to braided Leibniz rule:
    \[
    \mathcal{I}_{(\xi, \lambda)}^* = \ell_1^*(\mathcal{F}_A^*) - \frac{1}{2} \left( \ell_2^*(A, \mathcal{F}_A^*) - \ell_2^*(\mathcal{F}_A^*, A) \right) + \frac{1}{4} \ell_2^*(\bar{R}_\alpha A, \ell_2^*(\bar{R}_\alpha A, A)) = (0, 0)
    \]