L_∞-Algebras of Gravity and their Braided Deformations

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Topological Field Theory
and Deformation Quantization

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Outline

- Introduction/Motivation
- L_∞-algebras and classical field theories
- Einstein-Cartan-Palatini (ECP) gravity and its L_∞-algebras
- Noncommutative ECP gravity and braided L_∞-algebras

with M. Dimitrijević Ćirić, G. Giotopoulos & V. Radovanović

Nonassociative Gravity?

- In certain non-geometric flux compactifications of string theory, low-energy effective dynamics of closed strings may be described by noncommutative or even nonassociative deformations of gravity
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- Problems with naive definition of gauge transformations:

$$\delta^\star_\alpha A \; = \; \mathrm{d}\alpha + [\alpha, A]_\star \; = \; \mathrm{d}\alpha + \alpha \star A - A \star \alpha$$

  Nonassociativity obstructs closure of gauge algebra:

$$(\delta^\star_\alpha \, \delta^\star_\beta - \delta^\star_\beta \, \delta^\star_\alpha) A \; \neq \; \delta^\star_{[\alpha,\beta]_\star} A$$

L_∞-Algebras in Physics & Mathematics

- Higher spin gauge theories with field-dependent gauge parameters:

(Berends, Burgers & van Dam '85)

$$(\delta_\alpha\,\delta_\beta - \delta_\beta\,\delta_\alpha)\Phi \;=\; \delta_{C(\alpha,\beta,\Phi)}\Phi$$

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$$\delta_\alpha\Phi \;=\; \sum_n \ell_n(\alpha,\Phi^{n-1})$$

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- Deformation theory: Kontsevich's Formality Theorem based on L_∞-quasi-isomorphims of differential graded Lie algebras
- Any classical field theory with "generalized" gauge symmetries is determined by an L_∞-algebra, due to duality with BV–BRST (Hohm & Zwiebach '17; Jurčo, Raspollini, Sämann & Wolf '18)

L_∞-Algebras: Gravity/Noncommutative Gauge Theory

- L_∞-algebras of Einstein-Hilbert gravity: Requires perturbation about flat background, involves infinitely-many brackets
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- **In this talk:** Explain L_∞-algebra formulation of ECP gravity, define deformation with braided gauge symmetries, and then present braided L_∞-algebra determining noncommutative gravity

What is an L_∞-Algebra?

- Graded vector space: $V = \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots$, with graded exterior algebra $\Lambda_V = \wedge^\bullet(V[1])$ viewed as a free cocommutative coalgebra
- $L : \Lambda_V \longrightarrow \Lambda_V$ coderivation of degree $|L| = 1$, with $L^2 = 0$

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- Write $L^2 = 0$ in 'components' $L = \{\ell_n\}$ where
  $\ell_n : \wedge^n(V[1]) \longrightarrow V[1]$ with $|\ell_n| = 1$, or restoring original grading
  $\ell_n : \wedge^n V \longrightarrow V$ with $|\ell_n| = 2 - n$:

$$\ell_1(\ell_1(v)) = 0 \qquad (V, \ell_1) \text{ is a cochain complex}$$

$$\ell_1(\ell_2(v,w)) = \ell_2(\ell_1(v),w) \pm \ell_2(v,\ell_1(w)) \qquad \ell_1 \text{ is a derivation of } \ell_2$$

$$\ell_2(v,\ell_2(w,u)) + \text{cyclic} = (\ell_1 \circ \ell_3 \pm \ell_3 \circ \ell_1)(v,w,u) \qquad \text{Jacobi up to homotopy}$$

plus "higher homotopy Jacobi identities"

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- Dualizing gives graded commutative algebra derivation
  $Q = L^* : \Lambda_V^* \longrightarrow \Lambda_V^*$ with $|Q| = 1$, $Q^2 = 0$

L_∞-Quasi-Isomorphisms

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- Quasi-isomorphism is an equivalence relation on L_∞-algebras (contrary to dg Lie algebras)

Cyclic L_∞-Algebras

- Cyclic pairing $\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$ is non-degenerate, graded symmetric, bilinear and satisfies cyclicity:

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n) \rangle \ = \ \pm \langle v_1, \ell_n(v_0, v_2, \dots, v_n) \rangle$$

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- Cyclic L_∞-morphisms $\Psi : \Lambda_V \longrightarrow \Lambda_{V'}$ preserve cyclic pairings:

$$\langle \psi_1(v), \psi_1(w)\rangle' \ = \ \langle v, w\rangle$$

$$\sum_{i=1}^{n-1} \langle \psi_i(v_1, \dots, v_i), \psi_{n-i}(v_{i+1}, \dots, v_n)\rangle' \ = \ 0$$

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$$Q_{\rm BV}\xi \;=\; \ell_1(\xi) + \frac{1}{2}\,\ell_2(\xi,\xi) + \cdots$$

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- BV symplectic form (inducing antibracket) of degree -1 on V induces cyclic pairing of degree -3

$\cdots$	V_0	V_1	V_2	V_3	$\cdots$
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- V_{-k} encode 'higher gauge transformations' (ghosts-for-ghosts, etc.) for reducible symmetries

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- Action: $S \;=\; \frac{1}{2}\,\langle A, \ell_1(A)\rangle - \frac{1}{3!}\,\langle A, \ell_2(A, A)\rangle + \cdots$

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- Quasi-isomorphic L_∞-algebras give equivalent field theories

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Example: Chern-Simons Theory

- $\dim(M) = 3$, $\mathfrak{g}$ = quadratic Lie algebra with pairing $\mathrm{Tr}_{\mathfrak{g}}$
- Cochain complex = (de Rham complex) $\otimes\, \mathfrak{g}$: $V = \Omega^\bullet(M,\mathfrak{g})$
- Brackets: $\ell_1 = \mathrm{d}$, $\ell_2 = [-,-]_{\mathfrak{g}}$
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- Chern-Simons gauge theory is organised by a dg Lie algebra

Einstein-Cartan-Palatini Gravity (4d)

$$S \ = \ \int_M \mathrm{Tr}(e \wedge e \wedge R) \ = \ \int_M \varepsilon_{abcd} \left(e^a \wedge e^b \wedge R^{cd}\right)$$

- Fields: $e : TM \longrightarrow \mathcal{V}$ bundle isomorphism onto 'fake tangent bundle' $\mathcal{V}$ with Minkowski metric η, defines coframe $e \in \Omega^1(M, \mathcal{V})$

  $R \ = \ \mathrm{d}\omega + \frac{1}{2}\,[\omega, \omega] \in \Omega^2(M, P \times_{\mathrm{ad}} \mathfrak{so}(1,3))$ curvature of spin connection ω on associated principal $SO(1,3)$-bundle $P \longrightarrow M$

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- Bianchi identities: $\mathrm{d}^\omega T \;=\; R \wedge e \;\;,\;\; \mathrm{d}^\omega R \;=\; 0$

$$T \;=\; \mathrm{d}^\omega e \;=\; \mathrm{d}e + \omega \wedge e \;=\; \text{torsion of } \omega$$

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- **Note:** In contrast to Einstein-Hilbert formulation,
  ECP theory makes sense for degenerate coframes e
  (required for L_∞-algebra formulation)

L_∞-Algebra Picture of ECP Gravity (3d)

- Cochain complex: $V_0 \xrightarrow{\ell_1} V_1 \xrightarrow{\ell_1} V_2 \xrightarrow{\ell_1} V_3$
- Gauge transformations: $(\xi, \lambda) \in V_0 \ = \ \Gamma(TM) \times \Omega^0(M, \mathfrak{so}(3))$
- Physical fields: $(e, \omega) \in V_1 \ = \ \Omega^1(M, \mathbb{R}^3) \times \Omega^1(M, \mathfrak{so}(3))$
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- Noether identities: $(\Xi, \Lambda) \in V_3 \ = \ \Omega^1(M, \Omega^3(M)) \times \Omega^3(M, \mathbb{R}^3)$
- Differential: $\ell_1(\xi, \lambda) = (0, \mathrm{d}\lambda) \quad \ell_1(e, \omega) = (0, 0) \quad \ell_1(E, \Omega) = (0, \mathrm{d}\Omega)$

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- Higher brackets:

$$\ell_2((\xi_1, \lambda_1), (\xi_2, \lambda_2)) = ([\xi_1, \xi_2], -[\lambda_1, \lambda_2] + \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1)$$

$$\ell_2((\xi, \lambda), (e, \omega)) = (-\lambda \cdot e + \mathcal{L}_\xi e, -[\lambda, \omega] + \mathcal{L}_\xi \omega)$$

$$\ell_2((e_1, \omega_1), (e_2, \omega_2)) = -(2\,\omega_2 \wedge \omega_1, \omega_1 \wedge e_2 + \omega_2 \wedge e_1)$$

$+$ more ℓ_2 related to field equations and Noether identities

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$$\delta_{(\xi,\lambda)}(e,\omega) \;=\; (-\lambda\cdot e+\mathcal{L}_\xi e\,,\, \mathrm{d}\lambda-[\lambda,\omega]+\mathcal{L}_\xi\omega) \;=\; \ell_1(\xi,\lambda)+\ell_2((\xi,\lambda),(e,\omega))$$

  - Field equations: $\mathcal{F}_{(e,\omega)} \;=\; (R,T) \;=\; \ell_1(e,\omega)-\frac{1}{2}\,\ell_2((e,\omega),(e,\omega))$
  - Noether identities:

$$\begin{aligned}\mathcal{I}_{(\xi,\lambda)} \;&=\; \big(\mathrm{d}x^\mu\otimes\,\mathrm{Tr}(\iota_{\partial_\mu}e\wedge\mathrm{d}R-\iota_{\partial_\mu}\mathrm{d}e\wedge R)+(e\leftrightarrow\omega)\,,\,\mathrm{d}^\omega T-R\wedge e\big)\\ &=\; \ell_1(\mathcal{F}_{(e,\omega)})-\ell_2((e,\omega),\mathcal{F}_{(e,\omega)}) \;=\; (0,0)\quad (\textit{off-shell})\end{aligned}$$

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  - ECP action: $S \;=\; \langle(e,\omega)\,,\,\ell_1(e,\omega)+\ell_2((e,\omega),(e,\omega))\rangle$
  - Extend to $\langle -,-\rangle : V_0\times V_3\longrightarrow\mathbb{R}$ using gauge invariance and integration by parts; then cyclicity on V_0 implies Noether identities

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- Describes BV–BRST formulation of ECP gravity (with higher brackets for $d\geq 4$) (Cattaneo & Schiavina '17)

Application: Chern-Simons Gravity

- 3d gravity is equivalent to Chern-Simons theory with gauge algebra $\mathfrak{g} = \mathfrak{iso}(3) = \mathbb{R}^3 \rtimes \mathfrak{so}(3)$, $A = (e, \omega)$, $\mathcal{F}_A = (R, T)$ (Witten '88)

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- For invertible e, diffeos $\xi \in \Gamma(TM)$ are equivalent on-shell to gauge transfs by $(\tau_\xi, \lambda_\xi) = (\iota_\xi e, \iota_\xi \omega) \in \Omega^0(M, \mathfrak{g})$:

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- There is an (off-shell) cyclic L_∞-quasi-isomorphism $\{\psi_n\}$ with $\psi_n = 0$ for $n \geq 3$ from the Chern-Simons dg Lie algebra to the ECP dg Lie algebra

Braided Noncommutative Deformation

- Let $\mathcal{F} = \mathrm{f}^{\alpha} \otimes \mathrm{f}_{\alpha} \in U\Gamma(TM) \otimes U\Gamma(TM)$ be a Drinfel'd twist;

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- More generally, if $\mathcal{F}$ is a cochain twist, then $U_{\mathcal{F}}\Gamma(TM)$ is a quasi-Hopf algebra and $\mathcal{A}_\star$ is a nonassociative algebra

Braided Gauge Symmetry

- Braided Lie algebra $\Omega^0_\star(M,\mathfrak{so}(3))$: $[\lambda_1,\lambda_2]_\star \ := \ [-,-]\circ\mathcal{F}^{-1}(\lambda_1\otimes\lambda_2)$
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- Braided version of Noether's Second Theorem gives "braided" Noether identities *off-shell*, justifies interpretation of local braided symmetries as "gauge"

Braided L_∞-Algebras

- If $(V, \{\ell_n\})$ is a classical L_∞-algebra in the category of $U\Gamma(TM)$-modules, then $(V, \{\ell_n^\star\})$ is a braided L_∞-algebra in the category of $U_{\mathcal{F}}\Gamma(TM)$-modules, where

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- Cyclic pairing: $\langle -, - \rangle_\star \ := \ \langle -, - \rangle \circ \mathcal{F}^{-1}$

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\ell_2^\star((\xi,\lambda),(e,\omega)) &= (-\lambda\star e + \mathcal{L}^\star_\xi e, -[\lambda,\omega]_\star + \mathcal{L}^\star_\xi\omega) \\
\ell_2^\star((e_1,\omega_1),(e_2,\omega_2)) &= -(2\,\omega_2\wedge_\star\omega_1, \omega_1\wedge_\star e_2 + \bar{\mathrm{R}}^\alpha\omega_2\wedge_\star\bar{\mathrm{R}}_\alpha e_1)
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- Organises all dynamics of 3d noncommutative gravity:
  - Gauge symmetry and field equations as classically $(A = (e,\omega))$:

$$\delta^\star_{(\xi,\lambda)}A = \ell_1^\star(A) + \ell_2^\star((\xi,\lambda),A) \quad , \quad \mathcal{F}^\star_A = \ell_1^\star(A) - \tfrac{1}{2}\,\ell_2^\star(A,A)$$

  - Noether identities due to braided Leibniz rule:

$$\mathcal{I}^\star_{(\xi,\lambda)} = \ell_1^\star(\mathcal{F}^\star_A) - \tfrac{1}{2}\left(\ell_2^\star(A,\mathcal{F}^\star_A) - \ell_2^\star(\mathcal{F}^\star_A,A)\right) + \tfrac{1}{4}\,\ell_2^\star\left(\bar{\mathrm{R}}^\alpha A, \ell_2^\star(\bar{\mathrm{R}}_\alpha A, A)\right) = (0,0)$$