Universal Deformation Formulae

Pierre Bieliavsky

Universal Deformation Formula = tool that deforms objects (e.g. alegbras) from their symmetries.

Example

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Prop. : $\star^\alpha_\nu$ is a star product on M

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$$\mathcal{U}(\mathfrak{g})_F \ := \ (\mathcal{U}(\mathfrak{g})[[\nu]]\,,\,.\,,\, \Delta_F \ := \ F^{-1}\Delta F\ )$$

does (as a Hopf-algebra–quantum group).

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Non-formal version?
Seeking for non-formal star product on G

$$u \star_\theta v \ := \ \mu_0 \int_{G\times G} K_\theta(x,y)\ R_x^\star \otimes R_y^\star \,\mathrm{d}x\,\mathrm{d}y$$

where

$$R_x(g) \ = \ gx$$

$$[\theta \mapsto K_\theta] \ \in \ C^\infty(\mathbb{R}, \mathcal{D}'(G\times G))$$

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Then:

$$a \star_\theta^\alpha b := \frac{1}{\theta^{2n}}\, \mu_{\mathbb{A}} \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{\frac{i}{\theta} S(0,x,y)}\, \alpha_x(a) \otimes \alpha_y(b)\, \mathrm{d}x\, \mathrm{d}y$$

$\star_\theta^\alpha : \mathbb{A}^\infty \times \mathbb{A}^\infty \to \mathbb{A}^\infty$ Fréchet algebra

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Poisson structure: $\{f, g\} \ = \ \mu_0(\mathrm{d}\alpha_{\partial_p} \wedge \mathrm{d}\alpha_{\partial_q}(f \otimes g))$

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(iii) for all $x \in M$, $\exists$ μ-preserving homeomorphism $s_x : M \to M$ such that :

$$S(x, y, z) = -S(x, s_x(y), z) \qquad \forall y, z \in M.$$

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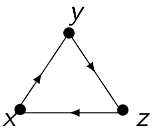

$\stackrel{\text{def.}}{=}\; S(x,y,z).$

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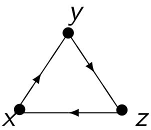
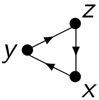


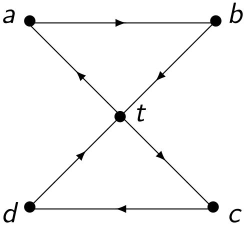
$$\overset{\text{def.}}{=} \; S(x,y,z).$$

$$u \star v(x) = \int_{M\times M} \exp i\big(\;\; \text{[graph]} \;\;\big)\; u(y)\; v(z)\; \mu(y)\; \mu(z)$$



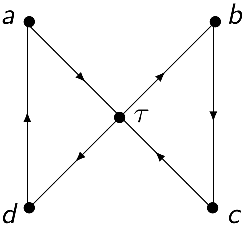
Associativity:

$\int_M \exp\ i($



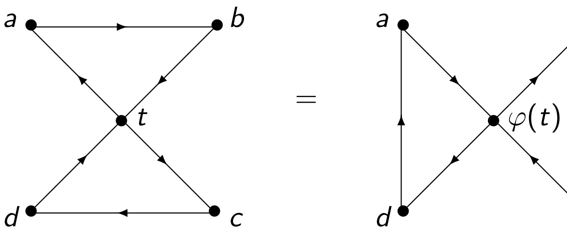
$)\ \mu(t) \quad =$

$\int_M \exp\ i($

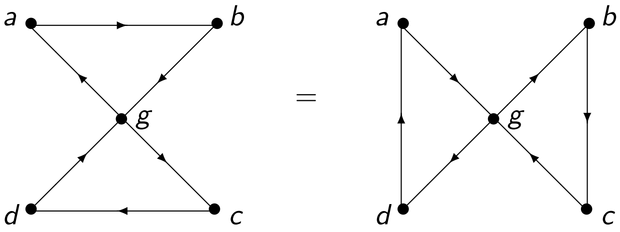


$)\ \mu(\tau)$

Volume preserving homeomorphism $\varphi : (M, \mu) \to (M, \mu)$ such that for all t,



S-barycenter: $g = g(a, b, c, d)$ such that



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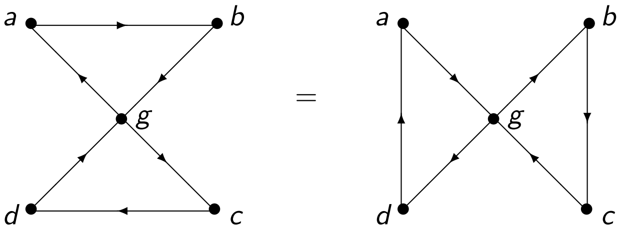

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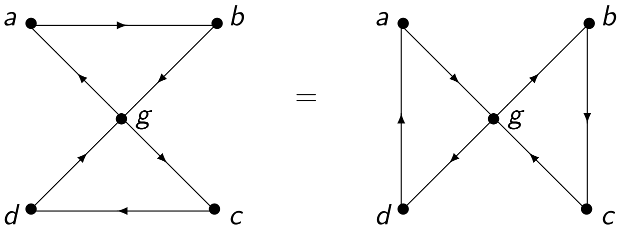

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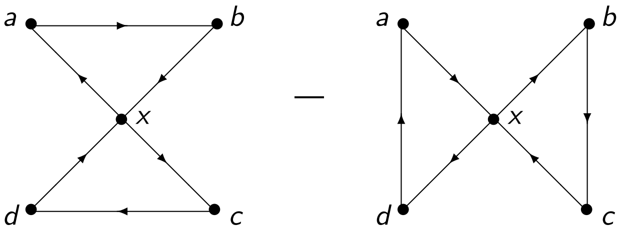

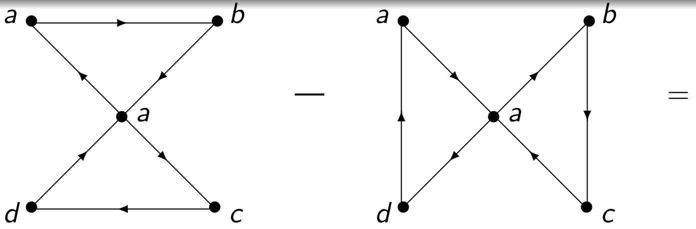
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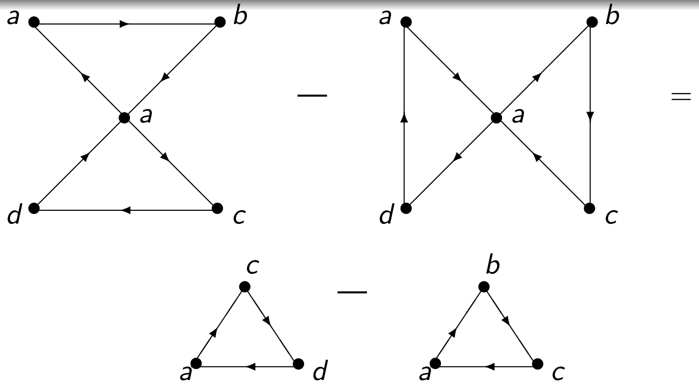
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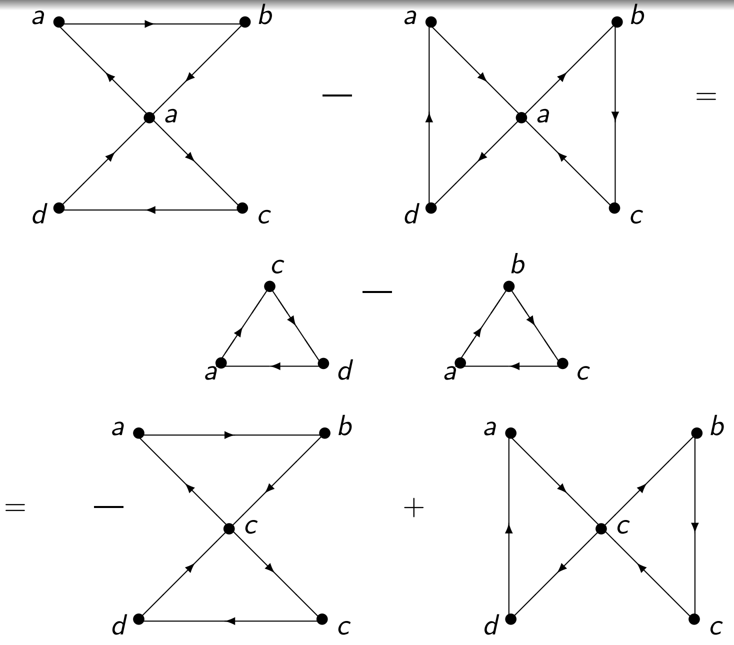
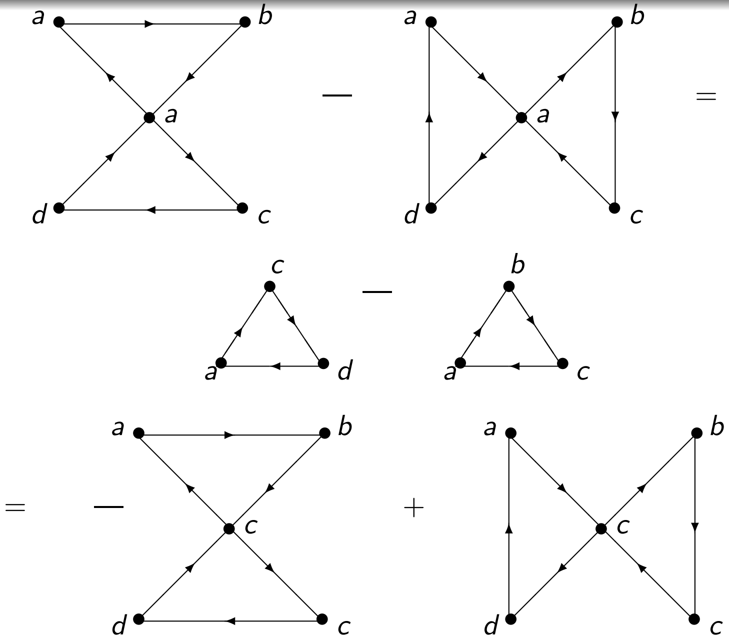
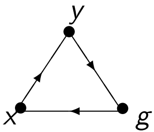




Any continuous path joining a to c contains such a point g.

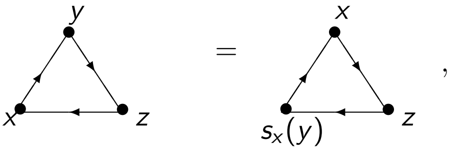
Fix S-barycenter g for $\{a, b, c, d\}$.



$\overset{\text{def.}}{=}$

.

Property (iii):



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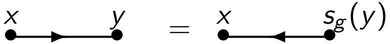

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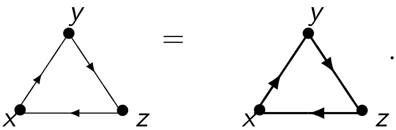
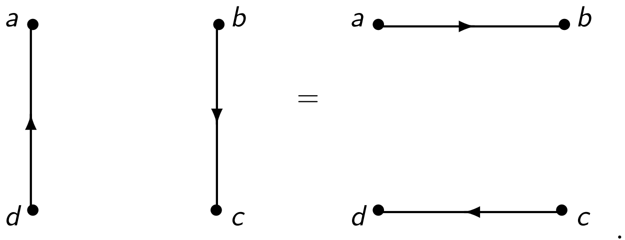
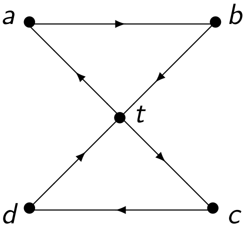


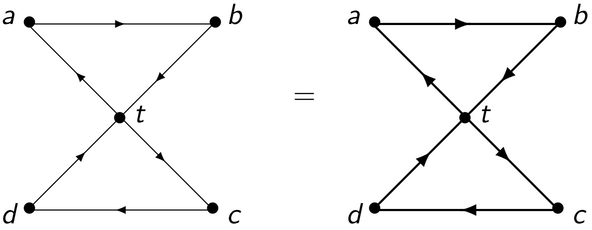
Barycentric property of g:

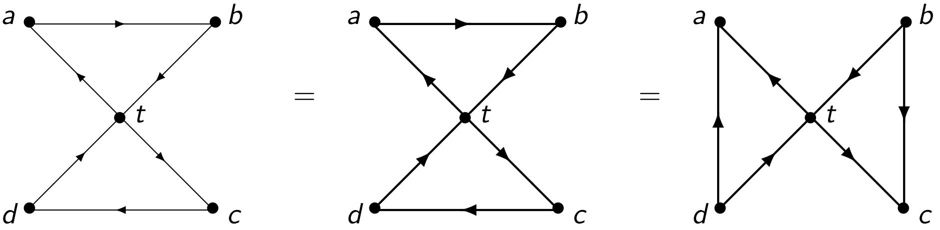




a
b
t
d
c

$=$


a
b
t
d
c
=
a
b
t
d
c


a
b
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=
a
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d
c
=
a
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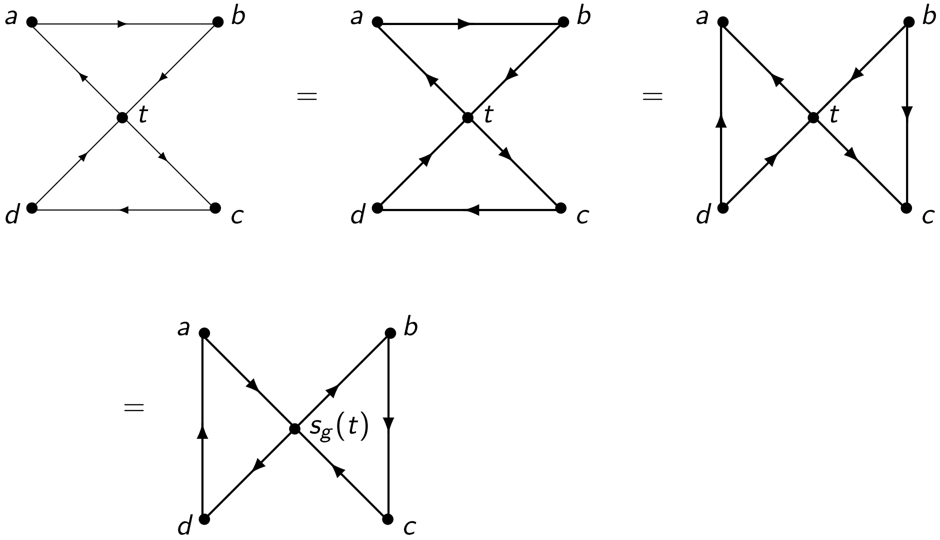
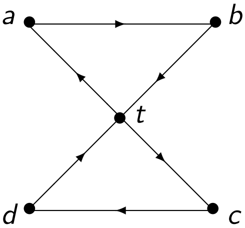
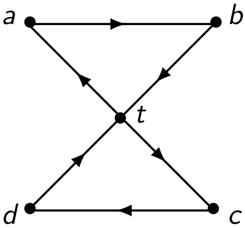
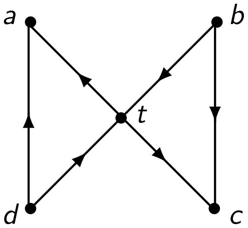

a
b
t
d
c
=
a
b
t
d
c
=
a
b
t
d
c
=
a
b
s_g(t)
d
c


a
b
t
d
c

$=$


a
b
t
d
c

$=$


a
b
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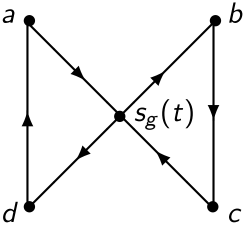
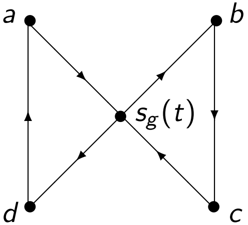
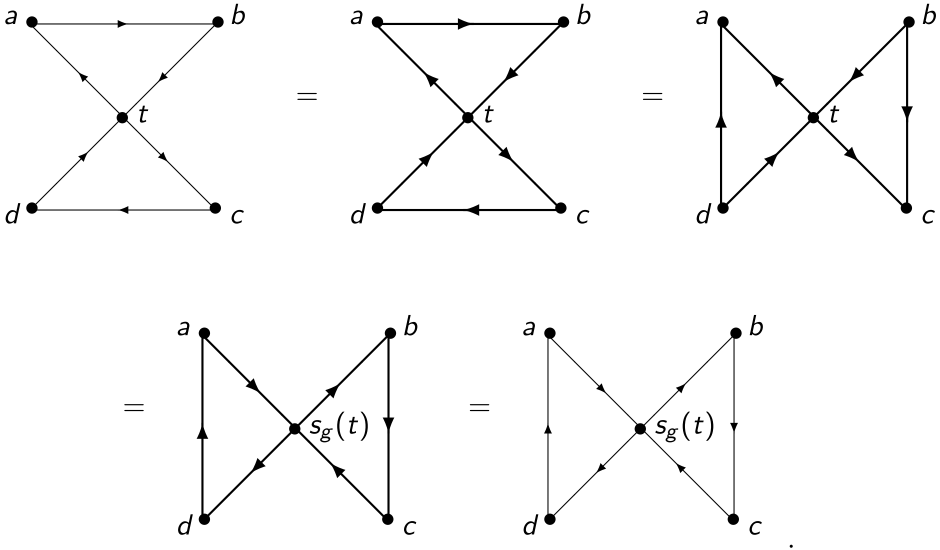

a
b
$s_g(t)$
d
c

$=$


a
b
$s_g(t)$
d
c

.



$$\varphi = s_g$$

Non-Abelian groups [Bieliavsky-Gayral, Mem. AMS. 15']

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$\mathbb{D} = L/K$ homogeneous complex bounded domain in $\mathbb{C}^n$

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$\Phi : \; G \times G \times G \to G \times G \times G : (x, y, z) \mapsto (\; \xi \; , \; s_z(\xi) \; , \; s_y s_z(\xi) \;)$ diffeo.

$$S(x,y,z) \ := \oint_{\Phi(x,y,z)} \omega$$

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$$K_\theta(x,y) \;=\; \frac{1}{\theta^{\dim G}} \left| Jac_\Phi(e,x,y) \right|^{1/2} \; e^{\frac{i}{\theta} \oint_{\Phi(e,x,y)} \omega}$$

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$$\Rightarrow$$

UDF actions of G on Fréchet or $C^\star$-algebras.

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$$\Rightarrow$$

UDF actions of G on Fréchet or $C^\star$-algebras.

Rem. Proof very different from Rieffel's case. It uses pseudo-differential operator theory, representation theory, symmetric spaces theory, wavelet analysis.

The formula

$$u \star_\theta v(x) \ := \ \frac{1}{\theta^{\dim G}} \int_{G\times G} |Jac_\Phi(x,y,z)|^{1/2} \ e^{\frac{i}{\theta}\oint_{\Phi(x,y,z)}\omega} u(y)\ v(z)\,\mathrm{d}y\,\mathrm{d}z$$

extends from $\mathcal{D}(G)$ to

$$\star_\theta \ : \ L^2(G) \times L^2(G) \to L^2(G)$$

as a G-equivariant associative Hilbert algebra structure:

Hilbert-Schmidt operators on irreducible unitary mass representation of G*.*

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THANK YOU VERY MUCH!