Topological quantum field theory and orbifolds

Nils Carqueville

Universität Wien

Motivation: quantum field theory

$$\textbf{spacetime} \xrightarrow{\quad\textbf{QFT}\quad} \textbf{algebra}$$

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$$\textbf{spacetime} \supset \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect} \subset \textbf{algebra}$$

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$$\textbf{spacetime} \supset \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \xrightarrow{\textbf{defect TQFT}} \mathrm{Vect} \subset \textbf{algebra}$$

Motivation: group representations

Let G be a group. A ***G*-representation** is a functor

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(Functoriality means $\rho(e) = \mathrm{id}_V$ and $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$.)

Topological quantum field theory

A **2-dimensional TQFT** is a symmetric monoidal functor

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Atiyah 1988

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orient. circles S^1 and surfaces with bdry./diffeom. vector spaces and linear maps

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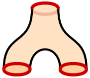

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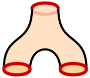

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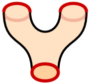

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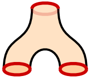

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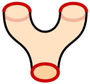

$$\longmapsto \quad \left(V \xrightarrow{\Delta} V \otimes V\right)$$

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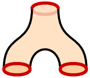
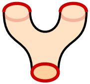
 $\longmapsto \left(V \otimes V \xrightarrow{\mu} V \right)$

 $\longmapsto \left(V \xrightarrow{\Delta} V \otimes V \right)$

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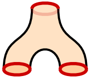
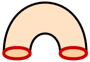
 $\circ$  $=$  $\longmapsto \left(\langle -, - \rangle : V \otimes V \xrightarrow{\mathrm{tr} \circ \mu} \mathbb{C} \right)$

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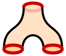
$S^1 \longmapsto V$ (**space of states**)

 $\longmapsto \big(\mu\colon V \otimes V \longrightarrow V\big)$ (associative **operator product**)

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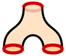

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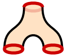 $\longmapsto \big(\mu\colon V \otimes V \longrightarrow V\big)$ (associative **operator product**)

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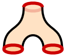 $\longmapsto \big(\mu\colon V \otimes V \longrightarrow V\big)$ (associative **operator product**)

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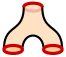

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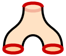 $$\longmapsto \big(\mu \colon V \otimes V \longrightarrow V\big)$$ (associative **operator product**)

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- Landau-Ginzburg models: $V = \mathbb{C}[x_1, \ldots, x_n]/(\partial_{x_1} W, \ldots, \partial_{x_n} W)$

Defect TQFT

A **2-dimensional defect TQFT** is a symmetric monoidal functor

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Davydov/Kong/Runkel 2011

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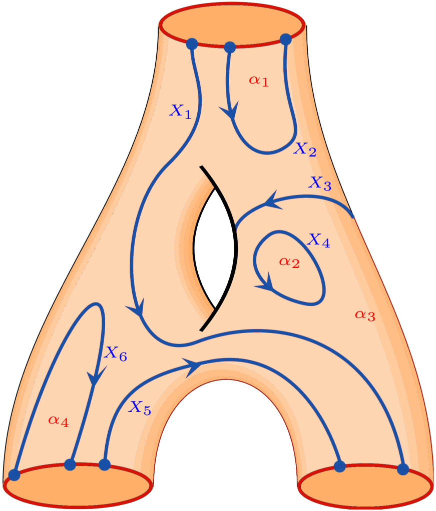

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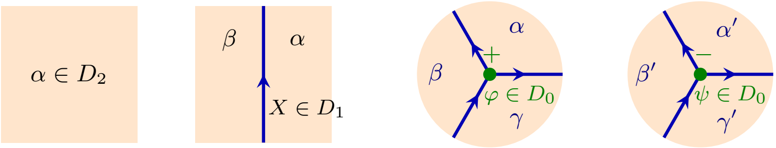
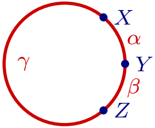
depending on **defect data** $\mathbb{D}$ consisting of:

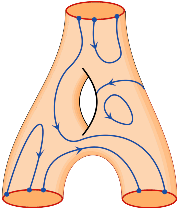
- set D_2 of bulk theories
- set D_1 of line defects
- set D_0 of junction fields



objects:



morphisms:



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Trivial defect TQFT $\mathcal{Z}^{\mathrm{triv}}$:

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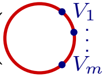
$\mathcal{Z}^{\mathrm{triv}}\Bigg($ 

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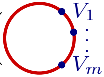
$\mathcal{Z}^{\mathrm{triv}}\Bigg($ 

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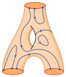
$\mathcal{Z}^{\mathrm{triv}}\Bigg($  $\Bigg) \stackrel{\mathrm{def}}{=}$ (evaluate 0- und 1-strata as string diagrams in Vect)

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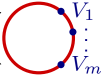
$\mathcal{Z}^{\mathrm{triv}}\Bigg($ 

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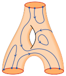
$\mathcal{Z}^{\mathrm{triv}}\Bigg($  $\Bigg) \stackrel{\text{def}}{=}$ (evaluate 0- und 1-strata as string diagrams in Vect)

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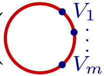
$\mathcal{Z}^{\mathrm{triv}}\Big($ 

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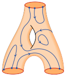
$\mathcal{Z}^{\mathrm{triv}}\Big($  $\Big) \stackrel{\mathrm{def}}{=}$ (evaluate 0- und 1-strata as string diagrams in Vect)

B-twisted sigma models:
Calabi-Yau manifolds and holomorphic vector bundles

Landau-Ginzburg models:
isolated singularities and homological algebra (more soon. . .)

State sum models

Input: Δ-separable symmetric Frobenius $\mathbb{C}$-algebra (A, μ, Δ)

Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

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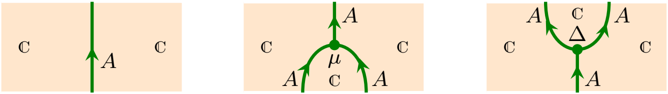

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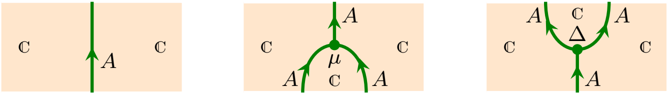

(3) Obtain $\Sigma^{t,A}$ in $\mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv}})$ and define $\mathcal{Z}_A^{\mathrm{ss}}(\Sigma) = \mathcal{Z}^{\mathrm{triv}}(\Sigma^{t,A})$

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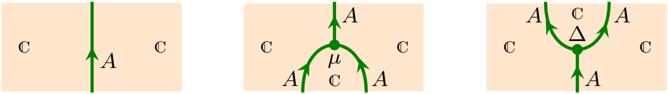

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Theorem. Construction yields TQFT $\mathcal{Z}_A^{\mathrm{ss}} \colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}$.

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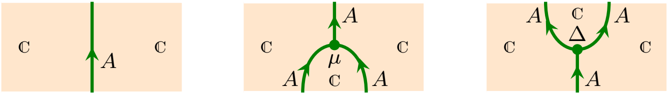

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Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

State sum models

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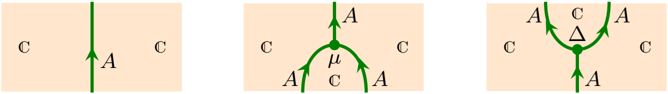

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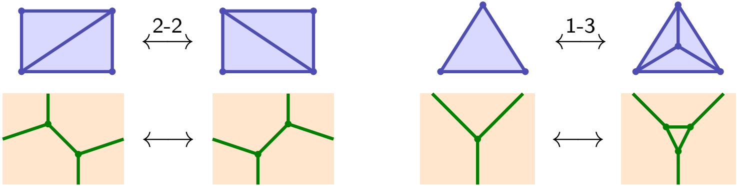

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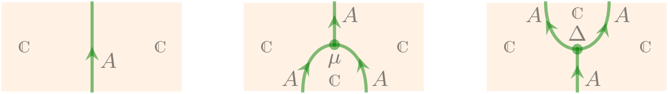

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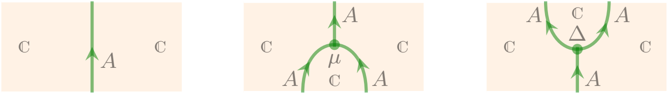

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No need to consider only algebras over $\mathbb{C}$!

Orbifolds

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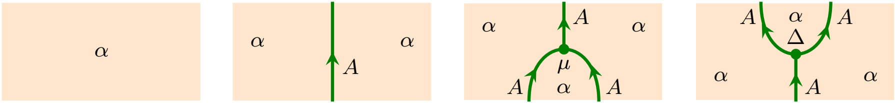

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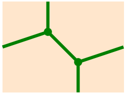
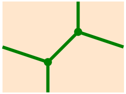
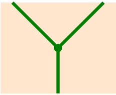
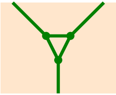
$\mathcal{Z}\Big($  $\Big) \overset{!}{=} \mathcal{Z}\Big($  $\Big)$ $\qquad$ $\mathcal{Z}\Big($  $\Big) \overset{!}{=} \mathcal{Z}\Big($  $\Big)$

Carqueville/Runkel 2012

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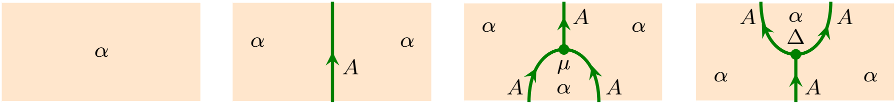

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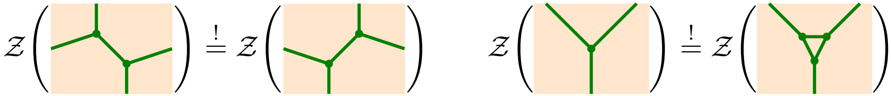

Definition & Theorem.
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$

Carqueville/Runkel 2012

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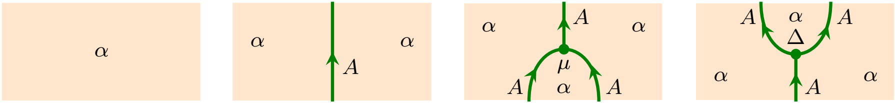

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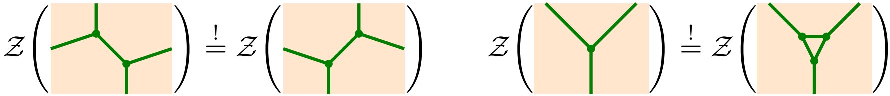

Definition & Theorem.
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$ = **$\mathcal{A}$-orbifold TQFT**

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Carqueville/Runkel 2012

Algebraic characterisation

Theorem.
2d defect TQFT $\mathcal{Z} \implies$ pivotal 2-category $\mathcal{B}_{\mathcal{Z}}$

Davydov/Kong/Runkel 2011

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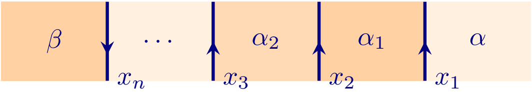

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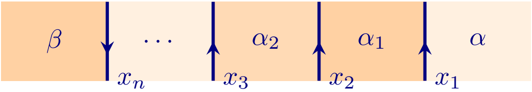
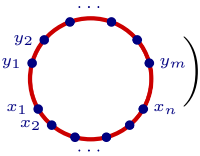


- 2-morphisms $=$ 'junction fields':

$\mathrm{Hom}(X, Y) = \mathcal{Z}\Bigg($  $\Bigg)$

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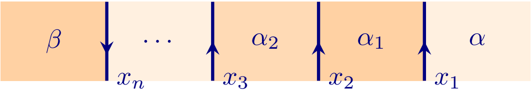

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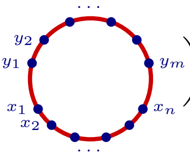

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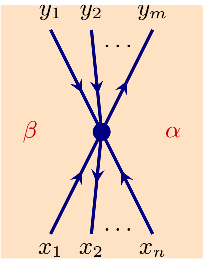

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Theorem.

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objects = bulk theories, 1-morphisms = defect lines, 2-morphisms = junction fields

Davydov/Kong/Runkel 2011

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Examples.

- **vector spaces**: BVect

  $*$, finite-dimensional $\mathbb{C}$-vector spaces, linear maps

Davydov/Kong/Runkel 2011, Carqueville 2016

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- **B-twisted sigma models**
  Calabi-Yau varieties, Fourier-Mukai kernels, RHom
- **A-twisted sigma models**
  symplectic manifolds, Lagrangian correspondences, Floer homology
- **Landau-Ginzburg models**
  isolated singularities, matrix factorisations
- **differential graded categories**
  smooth and proper dg categories, dg bimodules, intertwiners
- **categorified quantum groups**
  weights, functors $\mathcal{E}_i, \mathcal{F}_j \ldots$, string diagrams...

Davydov/Kong/Runkel 2011, Carqueville 2016

Algebraic characterisation of orbifolds

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Davydov/Kong/Runkel 2011

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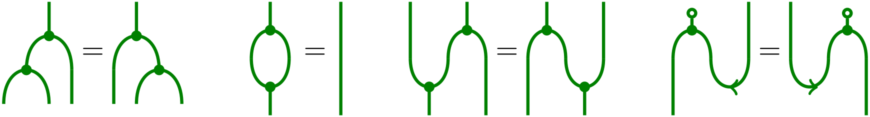

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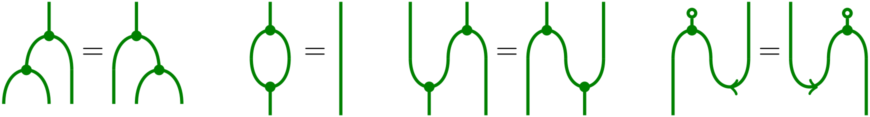

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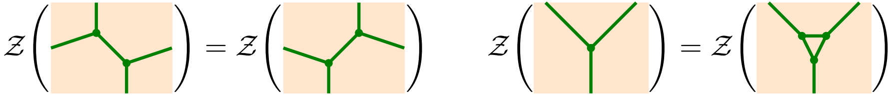

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Davydov/Kong/Runkel 2011

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Davydov/Kong/Runkel 2011, Fröhlich/Fuchs/Runkel/Schweigert 2009, Brunner/Carqueville/Plencner 2014

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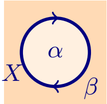
Orbifolds unify gauging of symmetry groups and state sum models.

Davydov/Kong/Runkel 2011, Fröhlich/Fuchs/Runkel/Schweigert 2009, Brunner/Carqueville/Plencner 2014

Orbifold equivalence: main idea

Let $X\colon \alpha \longrightarrow \beta$ be line defect such that 

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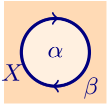
Let $X\colon \alpha \longrightarrow \beta$ be line defect such that 

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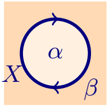
Let $X\colon \alpha \longrightarrow \beta$ be line defect such that 

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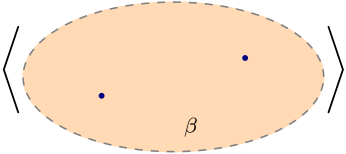

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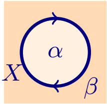
Let $X\colon \alpha \longrightarrow \beta$ be line defect such that  $\neq 0$ in correlators.

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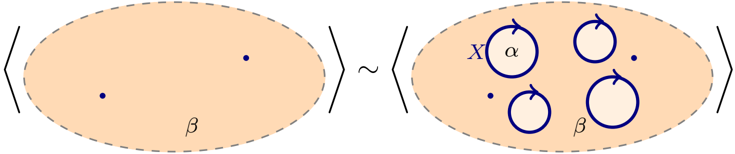

Carqueville/Runkel 2012

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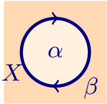
Let $X\colon \alpha \longrightarrow \beta$ be line defect such that 

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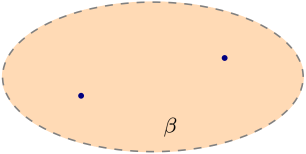
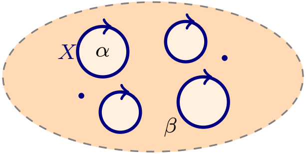
$\Big\langle$ 

$\Big\rangle \sim \Big\langle$ 

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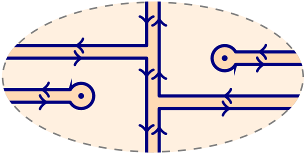
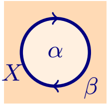
$= \Big\langle$  $\Big\rangle$

Orbifold equivalence: main idea

Let $X\colon \alpha \longrightarrow \beta$ be line defect such that 

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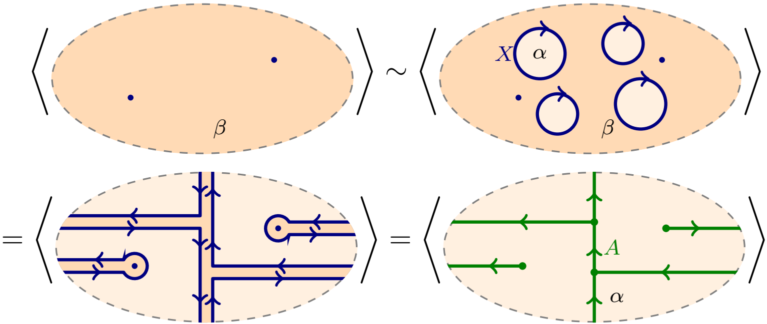

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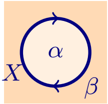
Let $X\colon \alpha \longrightarrow \beta$ be line defect such that 

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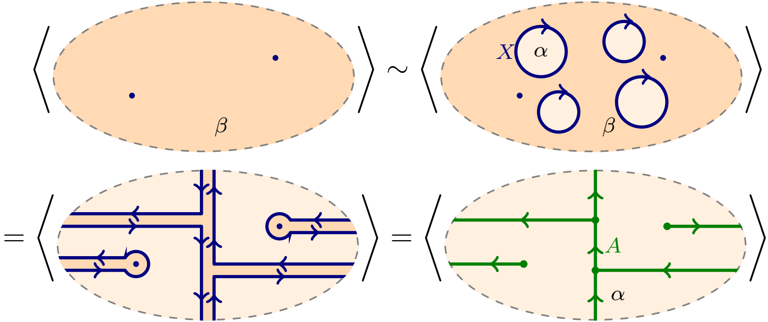

Theorem. (**orbifold equivalence** $\alpha \sim \beta$)

$$\big(\text{theory } \beta\big) \;\cong\; \big(A\text{-orbifold of theory } \alpha\big)$$

Carqueville/Runkel 2012

Orbifolds of Landau-Ginzburg models

Theorem. There is a pivotal 2-category $\mathcal{LG}$ with:

– objects = potentials $W \in \mathbb{C}[x_1, \dots, x_n]$

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Examples. $W_{\mathrm{A}_{n-1}} = x_1^n + x_2^2, \quad W_{\mathrm{D}_{n+1}} = x_1^n + x_1 x_2^2, \quad W_{\mathrm{E}_7} = x_1^3 + x_1 x_2^3$

Eisenbud 1980, Carqueville/Murfet 2012

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Examples. $\mathcal{D} = \left(\begin{smallmatrix} 0 & u^{n-i} \\ u^i & 0 \end{smallmatrix}\right)$ for u^n, $\mathcal{D} = \begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & y^2 & -x \\ x^2 & xy & 0 & 0 \\ xy^2 & -x^2 & 0 & 0 \end{pmatrix}$ for $x^3 + xy^3$

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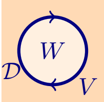
– 

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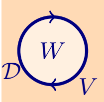
– 

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Theorem. (**Orbifold equivalences** in $\mathcal{LG}$)

$$\begin{array}{rcll} x^k + xy^2 & \sim & u^{2k} + v^2 & \qquad (\mathrm{D}_{k+1} \sim \mathrm{A}_{2k-1}) \\ x^3 + y^4 & \sim & u^{12} + v^2 & \qquad (\mathrm{E}_6 \sim \mathrm{A}_{11}) \\ x^3 + xy^3 & \sim & u^{18} + v^2 & \qquad (\mathrm{E}_7 \sim \mathrm{A}_{17}) \\ x^3 + y^5 & \sim & u^{30} + v^2 & \qquad (\mathrm{E}_8 \sim \mathrm{A}_{29}) \end{array}$$

Carqueville/Murfet 2012, Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013

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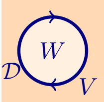
– 

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Carqueville/Murfet 2012, Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

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Theorem. $\overset{\text{(simple)}}{\mathrm{A}_{11}} \sim \overset{\text{(complicated)}}{\mathrm{E}_6}$ etc.

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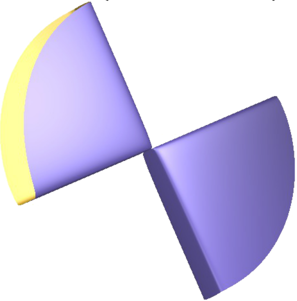

$\neq$

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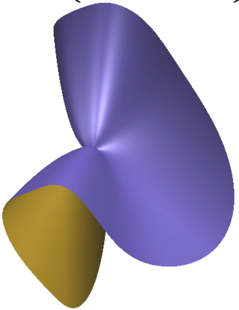

Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013, Recknagel/Weinreb 2017

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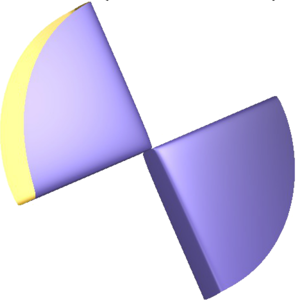

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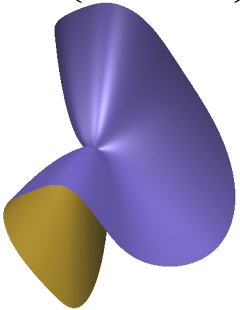

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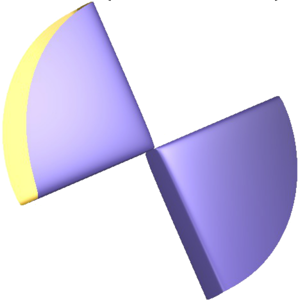

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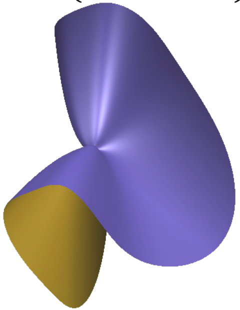

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Aside: Non-semisimple fully extended TQFTs

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Theorem.
For every potential W, the associated Landau-Ginzburg model $\mathrm{Bord}_{2,1} \longrightarrow \mathrm{Vect}$ can be lifted to a **fully extended TQFT**

$$\begin{aligned} \mathrm{Bord}_{2,1,0} &\longrightarrow \mathcal{LG} \\ \mathrm{pt}_+ &\longmapsto W \\ S^1 &\longmapsto \mathbb{C}[x_1,\dots,x_n]/(\partial_{x_1}W,\dots,\partial_{x_n}W) \end{aligned}$$

Carqueville/Montiel Montoya 2018

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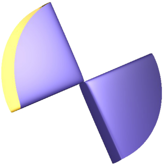
Carqueville/Montiel Montoya 2018

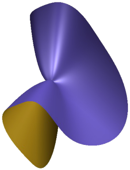
Summary so far

A_{11}



$\sim$

E_6



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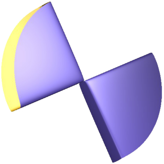

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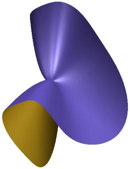
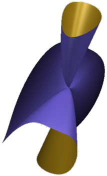
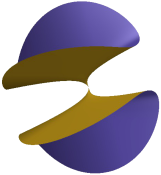


S_{11}



$\sim$

W_{13}



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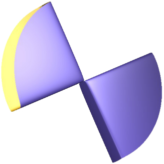

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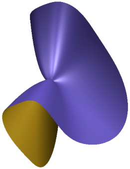

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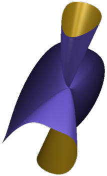

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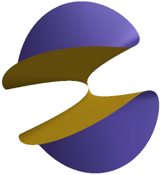

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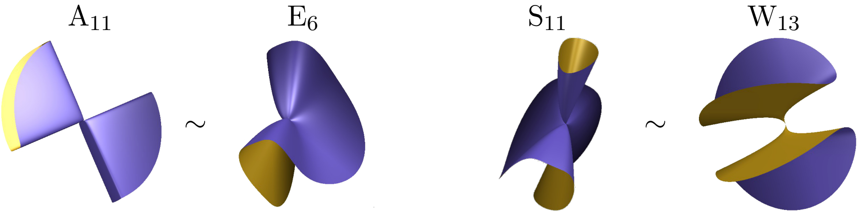

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2d orbifolds

- encode triangulation invariance in algebraic structure
- involve representation theory of algebras in 2-categories
- unify gauging of symmetry groups and state sum models
- **uncover new dualities**

The **orbifold construction** *can be generalised to n-dimensional defect TQFTs*

$$\mathcal{Z} \colon \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}$$

in any dimension $n \geqslant 1$.

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018

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Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018

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Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–18, Kapustin/Rozansky/Saulina 2009 + wip

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- sets D_j, whose elements decorate j-strata of bordisms
- rules how strata are allowed to meet (defined recursively via cones and cylinders)

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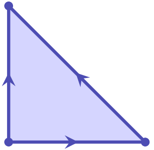
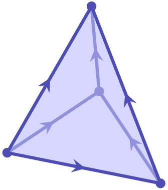
- **Rozansky-Witten theory** (conjecturally)
  - $D_3 = \{\text{holomorphic symplectic manifolds}\}$
  - $D_2 = \{\text{generalised Landau-Ginzburg models}\}$
  - $D_1 = \{\text{fibred matrix factorisations}\}$

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–18, Kapustin/Rozansky/Saulina 2009 + wip

Triangulations

standard n-simplex $\Delta^n := \left\{ \sum_{i=1}^{n+1} t_i e_i \;\middle|\; t_i \geqslant 0\,,\ \ \sum_{i=1}^{n+1} t_i = 1 \right\} \subset \mathbb{R}^{n+1}$

$\Delta^2 =$ 

$\Delta^3 =$ 

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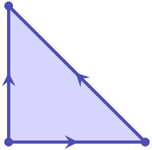
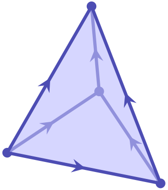
$\Delta^2 =$  $\Delta^3 =$ 

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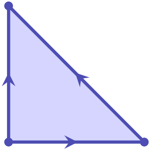
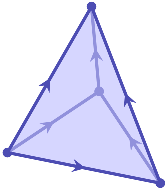
$\Delta^2 =$  $\Delta^3 =$ 

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triangulation of manifold M is simplicial complex C with homeomorphism $\varphi\colon |C| \xrightarrow{\cong} M$

(details for smooth, oriented, ...)

Pachner moves

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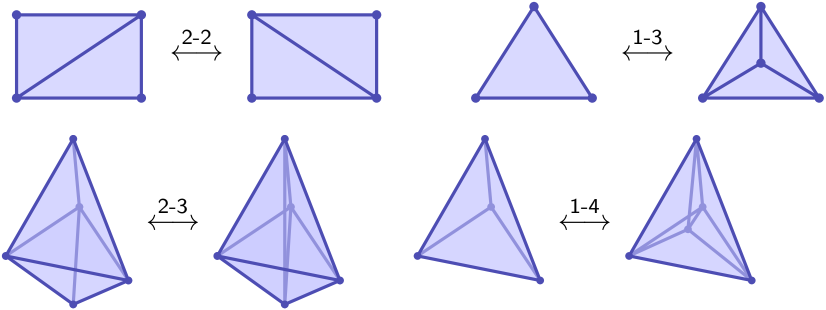

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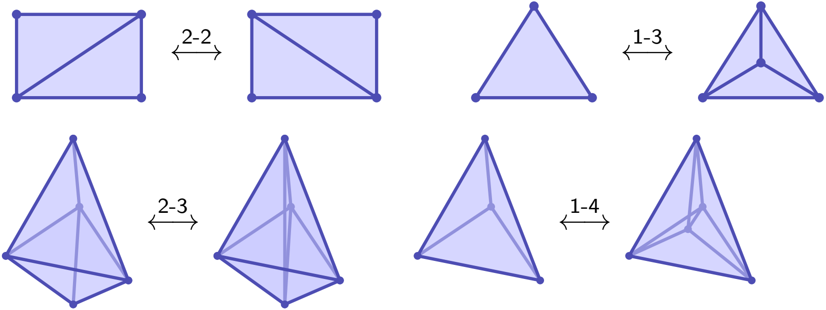

Theorem. If triangulated PL manifolds are PL isomorphic, then there exists a finite sequence of Pachner moves between them.

Pachner 1991

Orbifolds in any dimension n

An **orbifold datum** $\mathcal{A}$ for $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}$ consists of

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Carqueville/Runkel/Schaumann 2017

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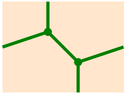
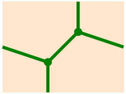
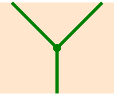
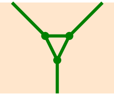
$\mathcal{Z}\Big($  $\Big) = \mathcal{Z}\Big($  $\Big)$ $\mathcal{Z}\Big($  $\Big) = \mathcal{Z}\Big($  $\Big)$

Carqueville/Runkel/Schaumann 2017

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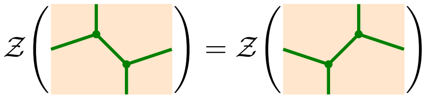
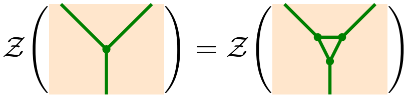


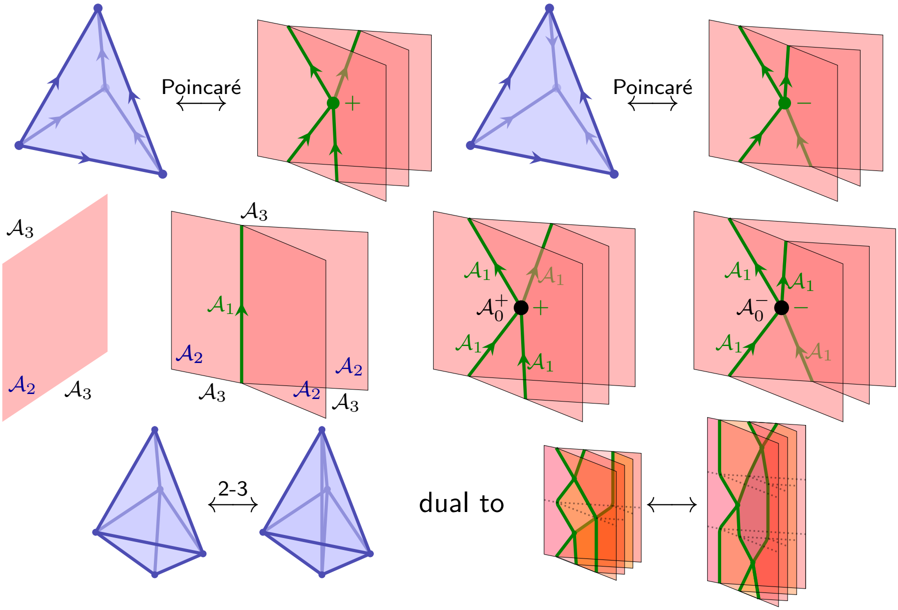


Definition & Theorem.
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$ = **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_n \longrightarrow \mathrm{Vect}$$

Carqueville/Runkel/Schaumann 2017

Orbifold datum $\mathcal{A}$ for $n = 3$



3d orbifolds

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3d defect TQFT $\mathcal{Z}$ $\Longrightarrow$ 3-category $\mathcal{T}_{\mathcal{Z}}$

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018

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Theorem.
Orbifolds of Reshetikhin-Turaev theories are Reshetikhin-Turaev theories.

Carqueville/Meusburger/Schaumann 2016, Carqueville/Runkel/Schaumann 2017–2018, C/Mulevičius/Runkel/Schaumann/Scherl 2020