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TWISTING PROCEDURE AND FORMALITY THEOREM

1. From DQ to Formality
2. Twisting procedure (a global approach)
3. Formality theorem
4. Symmetries ?

① FROM DQ TO FORMALITY

Formality = isomorphism between multivector fields & multidiff ops

Why ? Deformation quantization

Classical mechanics $\qquad M$ phase space

$C^\infty(M) \ni$ observables + Poisson bracket $\quad \{f,g\} = \underset{\displaystyle \Gamma^\infty(\Lambda^2 TM)}{\underset{\cap}{\pi}} (df, dg)$

$\Rightarrow (M, \pi)$ + Hamiltonian

$\Rightarrow C^\infty(M)$ associative commutative algebra + $\{\cdot,\cdot\}$

Quantum mechanics $\qquad \mathcal{H}$ Hilbert space

$B(\mathcal{H}) \ni$ observables $\Rightarrow$ non comm alg !

Q A - Given the classical algebra, can we find a suitable non commutative algebra which is a deformation of the classical one ? From the theory of formal deformation of associative algs $\Rightarrow$

we will comment

Def 1.1 (Star product)

A star product is

$$C^\infty(M)[[t]] \times C^\infty(M)[[t]] \longrightarrow C^\infty(M)[[t]]$$
$$(f, g) \longmapsto f \star g$$

i) $f \star g = f \cdot g + \sum_{k=1}^{\infty} t^k \, P_k \, (f, g)$

ii) associative

iii) P_k bidiff ops

iv) $f \star 1 = f = 1 \star f$

Remark 1.2

There are many quantization schemes available, the one proposed by DQ has conceptual advantages:

- its generality: other schemes uses the specific features of the classical system, DQ is universal as its requirements are minimal. Only Poisson algs! The hard part is actually prove $\exists$ (and classification) of $\star$ products [formality!]
- physical interpretation of observables fixed from beginning: they simply stay the same elements of the same underlying vector sp. Only the product law changes! Clear what tha Hamiltonian will be, the same as classical!

There is a price to pay ...

→ we asked $\star$ associative: non trivial, it results in a infinite chain of quadratic eqs for P_k operators [formality]

→ compatibility with classical system? we would like

$$P_1(f, g) - P_1(g, f) = i\{f, g\}$$

Automatically satisfied ! Thus we can identify

$t \mapsto \hbar$

Main difficulty : the deformation parameter correspond to a fundamental constant of nature , non zero and not dimensionless .

Makes no sense to speak of smallness of $\hbar$ $\Rightarrow$ convergence of the series $\star$ becomes important issue : convergence problem in DQ .

Quantization problem in the way we settled is : $\pi \longmapsto \star$?

(the converse is automatic)

We need to look at π and $\star$ in algebraic terms ...

⊙ star product $\equiv$ P_k which are in $\mathrm{Hom}(A \otimes A, A)$, $A = C^\infty(M)$

$\Rightarrow$ we look at $\mathrm{Hom}(A^{\otimes n}, A)$
(multidiff ops ↺)

⊙ Poisson bracket $\equiv$ π belongs to $\Gamma^\infty(\wedge^2 TM)$

$\Rightarrow$ we look at $\Gamma^\infty(\wedge^\cdot TM)$

Hochschild complex $\qquad A = C^\infty(M) \qquad m_0 : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$
standard product

$C^n(A) = \mathrm{Hom}(A^{\otimes n+1}, A) \qquad n \geq 0 \qquad$ GVS

$[\cdot,\cdot]_G \Rightarrow$ GLA $\qquad \partial := [m_0, \cdot]_G \Rightarrow$ DGLA

Obs. Associativity : $[m_0, m_0]_G = 0$

look at $P : A^{\otimes k} \to A$ polydiff ops. of rank $k+1$

$\Rightarrow$ subcomplex of $(C^\cdot(A), \partial, [\cdot,\cdot]_G)$
DGLA

$$D_{poly}^\cdot(M) = \bigoplus_{k=-1}^{\infty} D_{poly}^k(M)$$

<u>Theorem</u>[1.3] [HKR] $\qquad H^\cdot(D_{poly}(M)) = T^\cdot_{poly}(M)$

polyvector fields on M

In other words we have a map

$$T^\cdot_{poly}(M) \longrightarrow D^\cdot_{poly}(M) \qquad \text{which is a quis of complexes}$$

Enough ? No : it does not preserves the Lie structure, $[\![\cdot,\cdot]\!]$ Schouten $\not\mapsto [\cdot,\cdot]_G$. Important : they contain the info of Poisson and associativity. We want a DGLA morphism ...

<u>Theorem</u>[1.4] [Kontsevich]

$\exists$ quis of DGLA's $\qquad T^\cdot_{poly}(\mathbb{R}^d) \longrightarrow D^\cdot_{poly}(\mathbb{R}^d)$

Globalization : Dolgushev (2005) + Cattaneo

② . TWISTING PROCEDURE (A GLOBAL PROCEDURE)

Step back to abstract structures

- L_∞ - algebra $\quad (\mathcal{L}, Q)$ $\qquad L_\infty$ = Lie up to homotopy

  Q_n structures, Lie + ...

  $\left\{\begin{array}{l} \mathcal{L} \text{ graded vector sp.} \\ Q \text{ coderivation on coalgebra } S(\mathcal{L}) \text{ cogenerated by } \mathcal{L} \\ \quad + \text{ suitable properties} \end{array}\right.$

  $\boxed{\text{Ex. DGLA}}$

- $F : \mathcal{L} \to \tilde{\mathcal{L}}$ $\quad L_\infty$ - morphism if $\quad FQ = \tilde{Q}F$

  L_∞ - quis if $\quad F_1$ quis of complexes

Let $(\mathcal{L}, Q)$ a L_∞ - alg and $\pi \in \mathcal{L}^1$:

$\rightarrow$ we can twist $(\mathcal{L}, Q)$ by π and obtain a new L_∞ - alg $(\mathcal{L}, Q^\pi) =: \mathcal{L}^\pi$

$$Q^\pi(a) := \exp(-\pi) \vee Q(\exp(\pi) \vee a)$$

$\rightarrow$ $F : \mathcal{L} \to \tilde{\mathcal{L}}$ $\quad$ the image of π via F is denoted by π_F

- if π is MC element $\quad \Rightarrow \quad \pi_F$ is MC

  $d\pi + \frac{1}{2}[\pi, \pi] = 0$

• the twisted morphism/quis $F^{\pi} : \mathcal{L}_1^{\pi} \to \mathcal{L}_1^{\pi F}$ is still L_∞-morph/quis

Idea: we aim to use the twisting method to obtain a certain quis. between two L_∞-algs. It is often too hard to construct such a morphism directly but one can use twisting procedure. We construct the tools needed to perform the twisting in a global manner and apply the method to establish formality. More explicit:

we want $\quad T_{poly}(M) \xrightarrow{F} D_{poly}(M)$

we have $\quad T_{poly}(\mathbb{R}^d) \xrightarrow{K} D_{poly}(\mathbb{R}^d)$

Dolgushev cooks up (Fedosov) resolutions:

$$\begin{tikzcd}
T_{poly} \arrow[d, "quis"'] & & D_{poly} \arrow[d, "quis"] \\
\boxed{} \arrow[rr, dashed] & & \boxed{}
\end{tikzcd}$$

they are quisomorphic (apply K fiberwise) but wrt the wrong differential.

Twisting via MC element: locally and checks consistency on overlapping charts

⇒ the global quis is <u>NOT</u> described as a twist of another morphism

In 1801.08472 (joint with N. de Kleijn) we solve this problem by further resolving the Fedosov resolutions

Starting point:

$\mathfrak{L}$ L_∞-algebra $(\mathcal{M}, \varphi)$ L_∞-module

graded vector sp | φ coder on $S(\mathfrak{L})$-comodule $S(\mathfrak{L}) \otimes \mathcal{M}$ cogenerated by $\mathcal{M}$

Consider a resolution of $\mathcal{M}$

$$0 \to (\mathcal{M}, \varphi) \xrightarrow{F} (\mathcal{M}^0, \varphi^0) \xrightarrow{\partial^0} (M^1, \varphi^1) \longrightarrow \dots$$

i.e. M^i are L_∞-modules over $\mathfrak{L}$ and ∂^i are L_∞-morphisms

→ Resolution adapted MC elements = MC elements π of $\mathfrak{L}$ s.t.

$$0 \to H(M, \varphi_0^\pi) \xrightarrow{F_0^\pi} H(M, (\varphi^0)_0^\pi) \xrightarrow{\partial_0^\pi} H(M, (\varphi^1)_0^\pi) \to$$

induced complex

is acyclic

→ Given resolutions $F: \mathcal{M} \longrightarrow \mathcal{M}^0$ and $G: \mathcal{N} \to \mathcal{N}^0$ of L_∞-modules, a L_∞-morphism of resolutions $F \mapsto G$ is a series of L_∞-morphisms $\mathcal{U}: \mathcal{M} \to \mathcal{N}$, $\mathcal{U}^\cdot: \mathcal{M}^\cdot \to \mathcal{N}^\cdot$ s.t.

$$\begin{array}{ccccccc} 0 \to & \mathcal{M} & \longrightarrow & \mathcal{M}^0 & \longrightarrow & \mathcal{M}^1 & \longrightarrow \\ & u\downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{N} & \longrightarrow & \mathcal{N}^0 & \longrightarrow & \mathcal{N}^1 & \longrightarrow \end{array}$$

commutes

Theorem[2.1] [E. - de Kleijn]

Suppose $\mathcal{U}: F \to G$ is an L_∞-morphism of resolutions from a resolution the $\mathfrak{L}$-module $\mathcal{M}$ to a resolution of the $\mathfrak{L}$-module $\mathcal{N}$.

Suppose that $\pi \in MC(F) \cap MC(G)$ and $(\mathcal{U}^n)^\pi$ is a quis $\forall n \geq 0$

$\Rightarrow$ $\mathcal{U}^\pi$ is a quis.

Formality is a consequence!

③ FORMALITY THM

To see this we need

1. Resolutions of T_{poly} and D_{poly} ✓ by Dolgushev
2. L_∞-morphism of resolutions
3. MC element to twist

$$
\begin{tikzcd}
T_{poly} \arrow[d, "\text{quis DGLA}"'] & & D_{poly} \arrow[d, "\text{quis DGLA}"] \\
\mathcal{M} \arrow[d] \arrow[rr, "\mathcal{U}", "\text{from fiberwise of } K"'] & & \mathcal{N} \arrow[d] \\
\check{C}^0(\mathcal{M}) \arrow[d] \arrow[rr, dashed, "\mathcal{U}^0", "\text{from fiberwise application of } K + \text{resolutions}"'] & & \check{C}^0(\mathcal{N}) \arrow[d] \\
{} \arrow[rr, "\mathcal{U}^1"] & & {}
\end{tikzcd}
$$

wrong diff.

← resolutions : Čech complex has L_∞-structure

$\Rightarrow$ commutes

Take Γ resolution adapted MC
to both

constructed in a suitable way to get the correct differential !

: from Theorem 2.1 we obtain that

$$
\begin{tikzcd}
T_{poly} \arrow[d] & D_{poly} \arrow[d] \\
\mathcal{M} \arrow[r, "\mathcal{U}^\Gamma"'] & \mathcal{N}
\end{tikzcd}
$$

are all quis
$\Downarrow$
formality !

T_{poly} formal fiberwise polyvect fields =

bundle over M with fiber $T_{poly}(\mathbb{R}^d)$ associated with the principal bundle of general linear frames in TM

$\Omega(M, T_{poly})$ = differential forms with values in the bundle T_{poly}

$\circ$, $[\,,\,]$, ∂ $\Rightarrow$ DGLA

$$T_{poly} \xrightarrow[\text{quis}]{} (\Omega(M, T_{poly}), D)$$

↳ Fedosov differential

$$\check{C}(I, \Omega(M, T_{poly})) = \prod_{a \in I^i} \Omega(U_a, T_{poly})$$

$$\begin{array}{ccc} T_{poly} & & D_{poly} \\ \downarrow & & \downarrow \\ (\Omega(T_{poly}), D) & & (\Omega(D_{poly}), \partial + D) \end{array}$$

$$0 \xrightarrow{\ \mathcal{K}\ } \partial$$

$$D \xrightarrow{\ \mathcal{U}^\Gamma\ } \partial + D$$

$$(Q = 0 \Rightarrow Q^\Gamma = D)$$