

TWISTING PROCEDURE AND FORMALITY THEOREM

1. From DQ to Formality
2. Twisting procedure (a global approach)
3. Formality theorem
4. Symmetries ?

① FROM DQ TO FORMALITY

Formality = isomorphism between multivector fields & multidiff ops

Why ? Deformation quantization

Classical mechanics M phase space

$C^\infty(M) \ni$ observables + Poisson bracket $\{f, g\} = \pi (df, dg)$
 $\Gamma^\infty(\wedge^2 TM)$

$\Rightarrow (M, \pi) +$ Hamiltonian

$\Rightarrow C^\infty(M)$ associative commutative algebra + $\{\cdot, \cdot\}$

Quantum mechanics \mathcal{H} Hilbert space

$B(\mathcal{H}) \ni$ observables \Rightarrow non comm alg !

Given the classical algebra, can we find a suitable non commutative algebra which is a deformation of the classical one ? From the theory of formal deformation of associative algs \Rightarrow

\downarrow we will comment

Def 1.1 (Star product)

A star product is

$$\begin{aligned} C^\infty(M)[\hbar] \times C^\infty(M)[\hbar] &\longrightarrow C^\infty(M)[\hbar] \\ (f, g) &\longmapsto f \star g \end{aligned}$$

i) $f \star g = f \cdot g + \sum_{k=1}^{\infty} \hbar^k P_k(f, g)$

ii) associative

iii) P_k bidiff ops

iv) $f \star 1 = f = 1 \star f$

Remark 1.2

There are many quantization schemes available, the one proposed by DQ has conceptual advantages:

- its generality: other schemes use the specific features of the classical system, DQ is universal as its requirements are minimal. Only Poisson algs! The hard part is actually prove \exists (and classification) of \star products [formality!]
- physical interpretation of observables fixed from beginning: they simply stay ~~the~~ the same elements of the same underlying vector sp. Only the product law changes! Clear what the Hamiltonian will be, the same as classical!

There is a price to pay ...

→ we asked \star associative: non trivial, it results in a infinite chain of quadratic eqs for P_k operators [formality]

→ compatibility with classical system? we would like

$$P_1(f, g) - P_2(g, f) = i \{f, g\}$$

Automatically satisfied! Thus we can identify

$t \approx \hbar$

Main difficulty: the deformation parameter correspond to a fundamental constant of nature, not zero and not dimensionless.

Makes no sense to speak of smallness of $\hbar \Rightarrow$ convergence of the series \star becomes important issue: convergence problem in DQ .

Quantization problem in the way we settled is: $\pi \mapsto \star$?

(the converse is automatic)

We need to look at π and \star in algebraic terms ...

◦ star product $\equiv P_k$ which are in $\text{Hom}(A^{\otimes k}, A)$, $A = C^\infty(M)$

\Rightarrow we look at $\text{Hom}(A^{\otimes n}, A)$
(multidiff ops \downarrow)

◦ Poisson bracket $\equiv \pi$ belongs to $\Gamma^\infty(\wedge^2 TM)$

\Rightarrow we look at $\Gamma^\infty(\wedge^1 TM)$

Hochschild complex

$A = C^\infty(M)$

$m_0: C^\infty(M) \otimes C^\infty(M) \rightarrow C^\infty(M)$
standard product

$C^n(A) = \text{Hom}(A^{\otimes n+1}, A)$

$n \geq 0$ GVS

$[\cdot, \cdot]_G \Rightarrow$ GLA

$\partial := [m_0, \cdot]_G \Rightarrow$ DGLA

Obs - Associativity: $[m_0, m_0]_G = 0$

look at $P: A^{\otimes k} \rightarrow A$ polydiff ops. of rank $k+1$

\Rightarrow subcomplex of $(C(A), \partial, [\cdot, \cdot]_G)$
DGLA

$$D_{\text{poly}}^*(M) = \bigoplus_{k=-1}^{\infty} D_{\text{poly}}^k(M)$$

Theorem^{1.3} [HKR]

$$H^0(D_{\text{poly}}(M)) = T_{\text{poly}}(M)$$

polyvector fields on M

In other words we have a map

$$T_{\text{poly}}(M) \rightarrow D_{\text{poly}}(M) \quad \text{which is a quis of complexes}$$

Enough? No: it does not preserve the Lie structure, $[\cdot, \cdot]$ Schouten

$\rightarrow [\cdot, \cdot]_G$. Important: they contain the info of Poisson and associativity. We want a DGLA morphism...

Theorem^{1.4} [Kontsevich]

$$\exists \text{ quis of DGLA's} \quad T_{\text{poly}}(\mathbb{R}^d) \rightarrow D_{\text{poly}}(\mathbb{R}^d)$$

Globalization: Dolgushev (2005) (f. Cattaneo)

②. TWISTING PROCEDURE (A GLOBAL PROCEDURE)

Step back to abstract structures

• L_∞ -algebra (\mathcal{L}, Q) $L_\infty = \text{Lie up to homotopy}$

$\left\{ \begin{array}{l} \mathbb{Z} \text{ graded vector sp.} \\ Q \text{ codivision on coalgebra } S(\mathcal{L}) \text{ cogenerated by } \mathcal{L} \\ + \text{ suitable properties} \end{array} \right.$

Q_n structures, Lie + ...

EX - DGLA

• $F: \mathcal{L} \rightarrow \tilde{\mathcal{L}}$ L_∞ -morphism if $FQ = \tilde{Q}F$

L_∞ -quis if F_1 quis of complexes

Let (\mathcal{L}, Q) a L_∞ -alg and $\pi \in \mathcal{L}^1$:

\rightarrow we can twist (\mathcal{L}, Q) by π and obtain a new L_∞ -alg $(\mathcal{L}, Q^\pi) =: \mathcal{L}^\pi$

$$Q^\pi(a) := \exp(-\pi) \vee Q(\exp(\pi) \vee a)$$

$\rightarrow F: \mathcal{L} \rightarrow \tilde{\mathcal{L}}$ the image of π via F is denoted by π_F

• if π is MC element $\Rightarrow \pi_F$ is MC

$$d\pi + \frac{1}{2}[\pi, \pi] = 0$$

• the twisted morphism $F^\pi : \mathcal{L}^\pi \rightarrow \mathcal{L}^\pi$ is still L_∞ -morph/quis

Idea : we aim to use the twisting method to obtain a certain quis between two L_∞ -algs. It is often too hard to construct such a morphism directly but one can use twisting procedure.

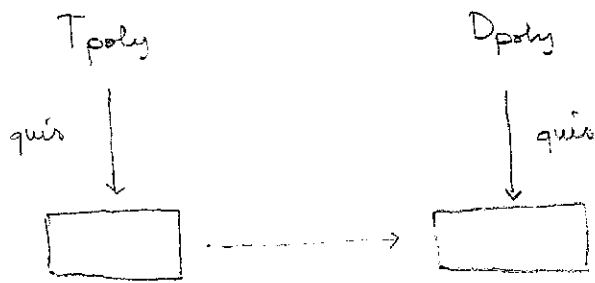
We construct the tools needed to perform the twisting in a global manner and apply the method to establish

formality - More explicit :

we want $T_{\text{poly}}(M) \xrightarrow{F} D_{\text{poly}}(M)$

we have $T_{\text{poly}}(\mathbb{R}^d) \xrightarrow{K} D_{\text{poly}}(\mathbb{R}^d)$

Do'gush'ev cooks up (Fedosov) resolutions -



they are quismorphic (apply K fibrewise) but with the wrong differential.

Twisting via MC element : locally and checks consistency on overlapping charts

\Rightarrow the global quis is NOT described as a twist of another morphism

J_n 1801.08472 (joint with N. de Kieijn) we solve this problem

by further resolving the Fedosov resolutions

Starting point :

\mathcal{L} L_∞ -algebra (\mathcal{M}, φ) L_∞ -module

graded vector sp |
 φ coder on $S(\mathcal{L})$ -
 comodule $S(\mathcal{L}) \otimes \mathcal{M}$
 generated by \mathcal{M}

Consider a resolution of \mathcal{M}

$$0 \rightarrow (M, \varphi) \xrightarrow{F} (M^0, \varphi^0) \xrightarrow{\partial^0} (M^1, \varphi^1) \rightarrow \dots$$

i.e. M^i are L_∞ -modules over \mathcal{L} and ∂^i are L_∞ -morphisms

→ Resolution adapted MC elements = MC elements π of \mathcal{L} s.t.

$$0 \rightarrow H(M, \varphi^\pi) \xrightarrow{F_0^\pi} H(M, (\varphi^0)^\pi) \xrightarrow{\partial_0^\pi} H(M, (\varphi^1)^\pi) \rightarrow \dots$$

induced complex

is acyclic

→ Given resolutions $F: \mathcal{M} \rightarrow \mathcal{M}^0$ and $G: \mathcal{N} \rightarrow \mathcal{N}^0$

of L_∞ -modules, a L_∞ -morphism of resolutions $F \mapsto G$

is a series of L_∞ -morphisms $U: \mathcal{M} \rightarrow \mathcal{N}$, $U^0: \mathcal{M}^0 \rightarrow \mathcal{N}^0$ s.t.

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{M} & \rightarrow & \mathcal{M}^0 & \rightarrow & \mathcal{M}^1 \rightarrow \\ & & \downarrow U & & \downarrow & & \downarrow \\ & & \mathcal{N} & \rightarrow & \mathcal{N}^0 & \rightarrow & \mathcal{N}^1 \rightarrow \end{array}$$

commutes

Theorem 2.1 [E.-deKleijn]

Suppose $U: F \rightarrow G$ is an L_∞ -morphism of resolutions from a resolution of the L -module \mathcal{M} to a resolution of the L -module \mathcal{N} .

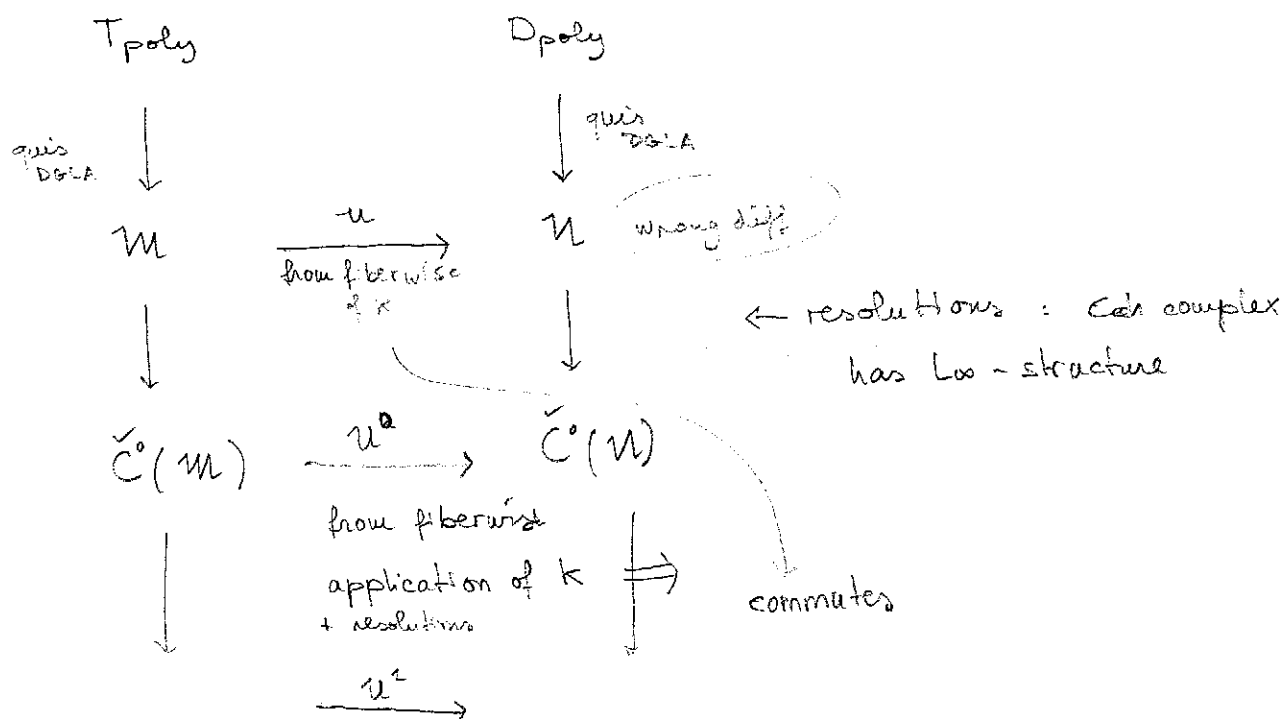
Suppose that $\pi \in MC(F) \cap MC(G)$ and $(U^0)^\pi$ is a quis $\forall n \geq 0$
 $\Rightarrow U^\pi$ is a quis.

Formality is a consequence!

③ FORMALITY THM

To see this we need

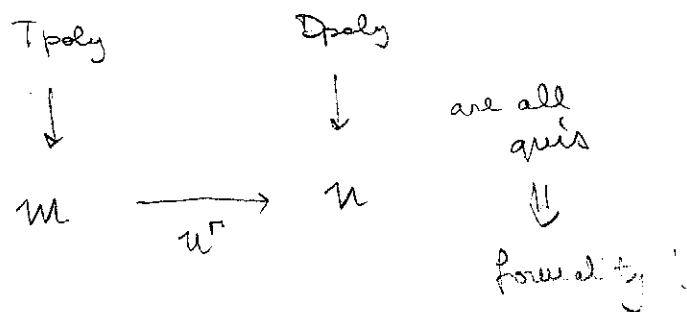
1. Resolutions of T_{poly} and D_{poly} ✓ by Dolgushev
2. L_∞ -morphism of resolutions
3. MC element to twist



Take Γ resolution adapted MC

\downarrow
 constructed in a suitable way to get the correct differential!

from Theorem 2.1 we obtain that



T_{poly} formal fiberwise polyvect fields =

bundle over M with fiber $T_{\text{poly}}(\mathbb{R}^d)$ associated ~~to~~
with the principal bundle of general linear frames in TM

$\Omega(M, T_{\text{poly}})$ = differential forms with values in the bundle T_{poly}

$0, \mathbb{I}, \mathbb{I} \Rightarrow \text{DGLA}$

$T_{\text{poly}} \xrightarrow{\text{quib}} (\Omega(M, T_{\text{poly}}), D)$
↳ Fedosov differential

$$\check{C}(\mathbb{I}, \Omega(M, T_{\text{poly}})) = \prod_{a \in \mathbb{I}^k} \Omega(U_a, T_{\text{poly}})$$

$$\begin{array}{ccc} T_{\text{poly}} & & D_{\text{poly}} \\ \downarrow & & \downarrow \\ (\Omega(T_{\text{poly}}), D) & & (\Omega(D_{\text{poly}}), \partial + D) \end{array}$$

$$0 \rightarrow k \rightarrow \partial$$

$$0 \rightarrow \mathcal{U}^r \rightarrow \partial + D$$

$$(Q = 0 \Rightarrow Q^r = D)$$