Homotopy Algebras in SFT

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joint work with
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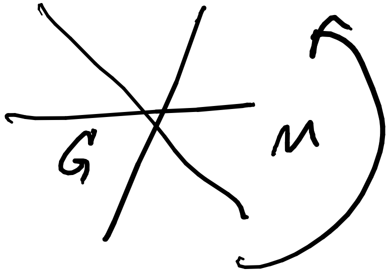
Cor

$2(G-1)+n>2$

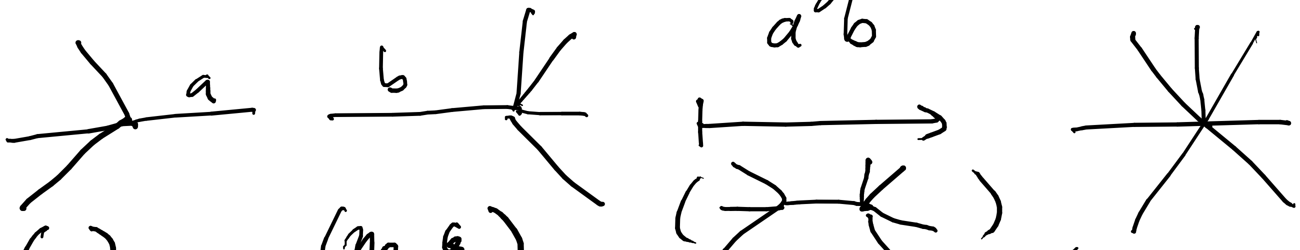


$\rho \in \Sigma_n$

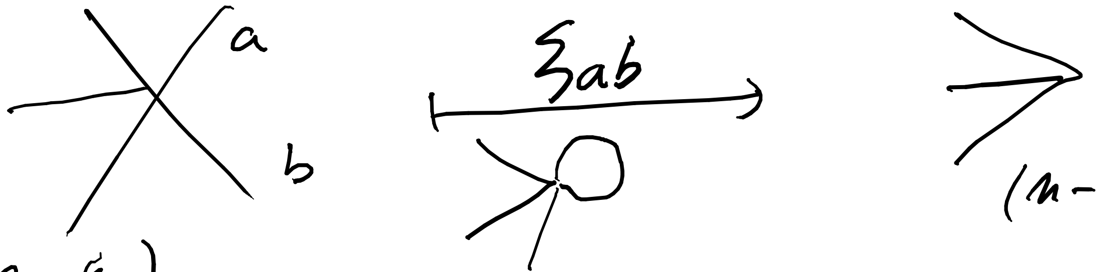
skeletal version

(non-skeletal $\{1,\dots,n\} \to C$ set $|C| = n$)

operations



(n_1, g_1) (n_2, g_2) $\xrightarrow{a \circ b}$ (n_1+n_2-2, g_1+g_2)



(n, g) $\xrightarrow{\xi_{ab}}$ $(n-2, G+1)$

Modular operad

$\mathcal{P}(\ast_g\, n)$ – collection of d.g.v.s

\+ morphisms deg = 0

$$\mathcal{P}(\ast) \xrightarrow{\mathcal{P}(\rho)} \mathcal{P}(\rho(\ast)) \qquad \Sigma\text{-action}$$

$$\mathcal{P}(\ast_a) \otimes \mathcal{P}({}_b\ast) \xrightarrow{{}_a\circ_b} \mathcal{P}(\ast)$$

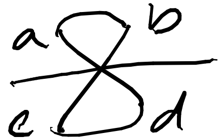
$$\mathcal{P}(\ast^{a\,b}) \xrightarrow{\xi_{ab}} \mathcal{P}(\ast)$$

such that:

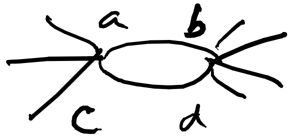
1. $\quad a \circ_b ,\ \xi_{ab}$ are Σ-equivariant

2. $\quad a \circ b \, (x \otimes y) = (-1)^{|x||y|} \, b \circ a \, (y \otimes x)$

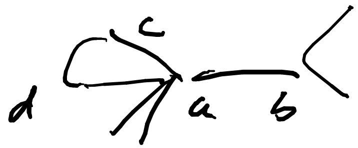
3. $\quad \xi_{ab} \xi_{cd} = \xi_{cd} \xi_{ab}$



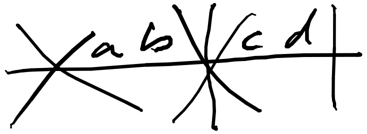
4. $\quad \xi_{ab} \, c \circ d = \xi_{cd} \, a \circ b$



5. $\quad a \circ b \, (\xi_{cd} \otimes 1) = \xi_{cd} \, a \circ b$



6. $\quad a \circ b \, (1 \otimes c \circ d) = c \circ d \, (a \circ b \otimes 1)$

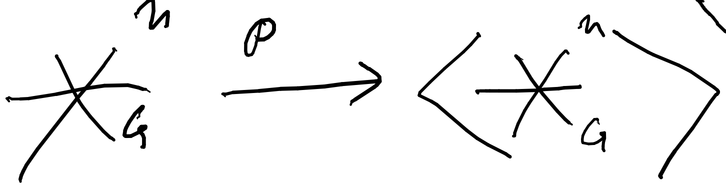


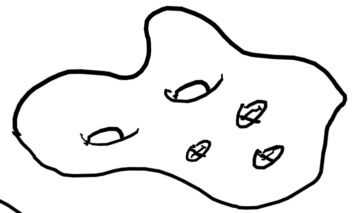
$G = 0$ + forgetting ξ_{ab}

cyclic operad

Adjoint functor $\qquad$ modular envelope

Ex. 1) Mod$(\mathcal{C}om^c)$



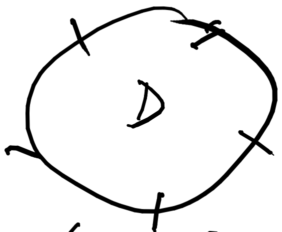


$G = g$

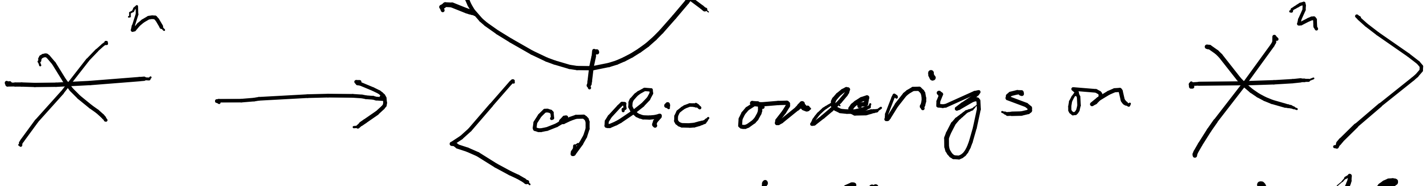
1-dim v.s. in degree 0
differential $d = 0$

Σ-action trivial $a \circ_b$, ξ_{ab} – obvious

2) $\mathcal{A}ss^c$

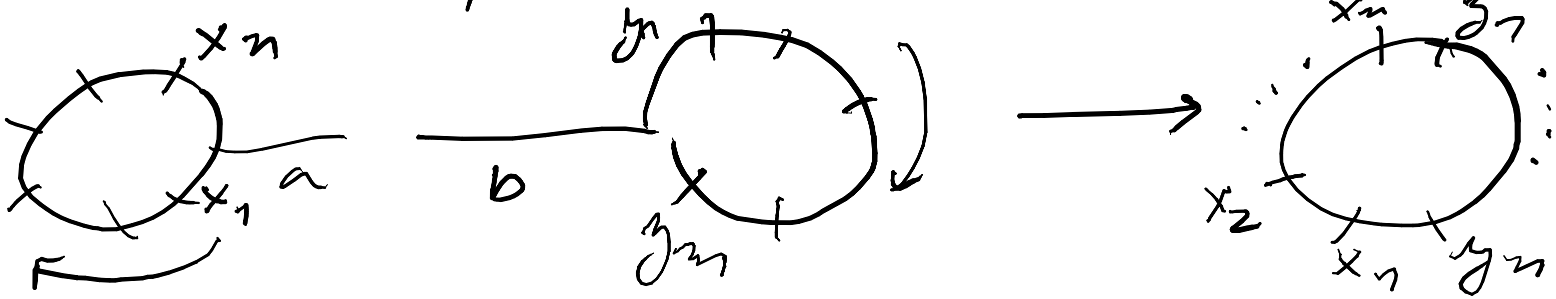


$G = 0$

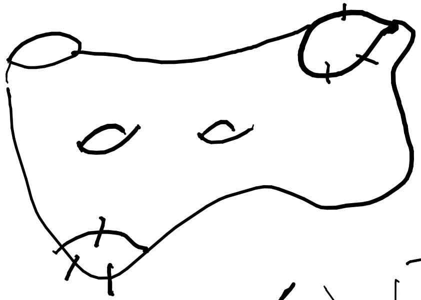


Σ-action – permutations of points on ∂D

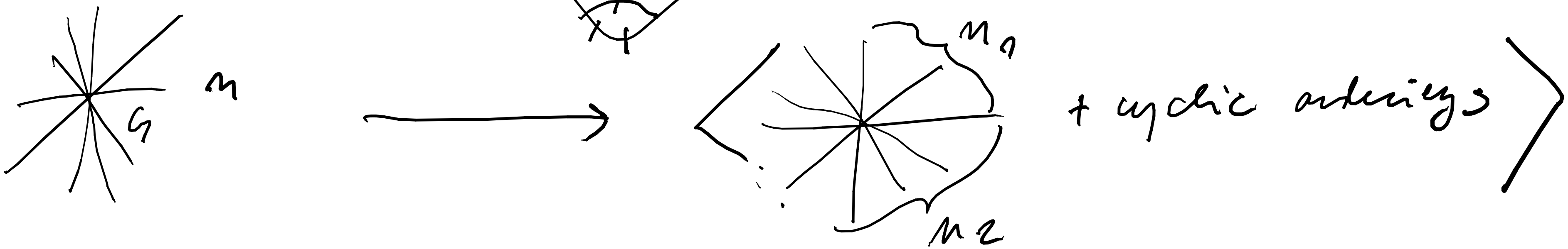
$a \circ_b$



3. $\mathcal{M}od(\mathcal{A}ss^c)$



$G = 2g + b - 1$



$a \circ_b$ sewing two surfaces

ξ_{ab} (2 kinds) – self-sewing of surface within the same boundary component

– self-sewing using point on different boundary components

4. Quantum open-closed

2-colored, combination of 1. and 3.

5. super versions

e.g. Mod (com^c $N=1$) type II

4-colored NS-NS, NS-R, R-NS, R-R

Odd modular operads

$a \circ b \quad \xi_{ab}$ are of degree $\underline{\underline{1}}$

$\Downarrow$

modification of axioms

$$\xi_{ab}\,\xi_{cd} = \ominus\,\xi_{cd}\,\xi_{ab}$$
$$\xi_{ab}\; c \circ d = \ominus\,\xi_{cd}\; a \circ b$$
$$\vdots$$

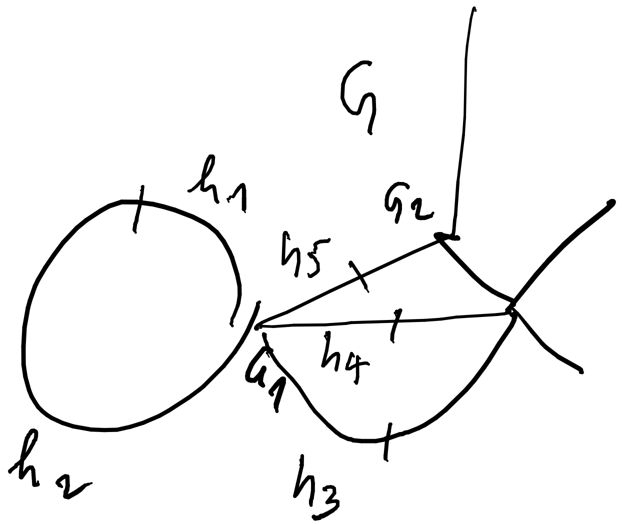
Ex. 1) $\mathrm{End}\,V$ $\qquad V$ - d.g. (super) v.s.

ω - deg -1 symplectic form, compatible with d

$\mathcal{P}(n, G) = \mathrm{Hom}(V^{\otimes n}, \mathbb{C}) =: \mathcal{E}_V(n, G)$ $\qquad \Sigma$ - permutation of inputs

$a \circ b\,,\ \xi_{ab}$ contracting inputs with ω

Ex. 2 Feynman transform $\mathcal{F}\mathcal{P}$ of a modular operad $\mathcal{P}$



$G = \sum G_i + \#\,\text{loops}$

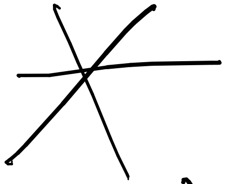
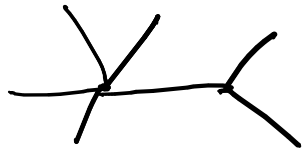
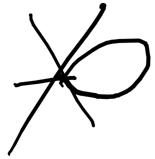
decorated by

$$\left(p_1 \otimes p_2 \otimes p_3 \right) \otimes \left(\uparrow e_1 \wedge \ldots \wedge \uparrow e_5 \right)$$

$$p_1 \in \mathcal{P}\left(G_1 \; h_1, h_2, h_3, h_4, h_5 \right)^{\#}$$

$(a \circ b)_{\mathcal{F}\mathcal{P}}$ $\quad (\xi_{ab})_{\mathcal{F}\mathcal{P}}$ grafting of graps and attaching edges, resp.

∂_{FP} – adding edge + modifying decorations using $(a \circ b)^{\#}$ and $\xi_{ab}^{\#}$

∂_{FP}  $=$ $\sum\limits_{\substack{n_1+n_2=n \\ G_1+G_2=G}}$  $+$ 

(n, G) $\qquad$ $G-1$

$G=0$, ξ_{ab} – trivial – cyclic cobar constr.

Algebra over Feynman transform

Morphism of odd modular operads

$$\mathcal{F}\mathcal{P} \longrightarrow \mathrm{End}\, V$$

Barannikov:

$\Longleftrightarrow$ following data

$$\{ m(C,G) \in (\mathcal{P}(C,G) \otimes \mathcal{E}_V(C,G))^{\Sigma} \}$$

$$\underbrace{(d_{\mathcal{E}_V} - d_{\mathcal{P}})}_{d} m(C,G) = \underbrace{(\xi_{ab})_{\mathcal{P}} \otimes (\xi_{ab})_{\mathcal{E}_V}}_{\Delta} m(C \sqcup \{a,b\}, G-1) +$$

$$\underbrace{\frac{1}{2} \sum_{\substack{C_1 \sqcup C_2 = C \\ G_1 + G_2 = G}}}_{} \underbrace{\Big((a \circ b)_{\mathcal{P}} \otimes (a \circ b)_{\mathcal{E}_V} \Big)}_{\{\,,\,\}} m(C_1 \sqcup \{a\}, G_1) \otimes m(C_2 \sqcup \{b\}, G_2)$$

(NC) BV-algebra on V

$$\left((\mathcal{P} \otimes \mathcal{E}_V)^{\Sigma}, d, \Delta, \{\cdot,\cdot\} \right)$$ d.g. Lie algebra

$d + \Delta$ - differential

$\{\cdot,\cdot\}$ - Lie bracket

<u>Action</u> (Interaction part) $\qquad \deg S = 0$

$$S_{int} := \sum_{n,g} \hbar^g m(n,g)$$ solution to the BV QME

$$dS_{int} + \hbar \Delta S_{int} + \frac{1}{2}\{S_{int}, S_{int}\} = 0$$

<u>Theorem</u> (Barannikov)

Algebras over $\mathcal{F}\mathcal{P}$ $\Longleftrightarrow$ solutions to QME

- Generalization $\mathcal{E}_V \to$ any odd modular operad
- Action in physics terms

a_i - basis of V $\qquad a_I = a_{i_1} \otimes \dots \otimes a_{i_n}$

ϕ^i - dual basis $\qquad \phi_I = \phi^{i_1} \otimes \dots \otimes \phi^{i_n}$

$$\left(\mathcal{P}(n,G) \otimes \mathcal{E}_V(n,G)\right)^{\Sigma_n} \cong \mathcal{P}(n,G) \otimes_{\Sigma_n} V^{\# \otimes n}$$

S_{int} - BV action ; Δ, $\{\cdot,\cdot\}$ - BV-operation, $\phi = a_i \otimes \phi^i$ string field

Ex. 1 $\mathrm{Mod}(\mathrm{Com}^c)$ $\qquad G = g$ (or $G = 2g + \frac{n}{2} - 1$)

$$\mathcal{P}(n,G) \otimes_{\Sigma_n} V^{\# n} \cong S^n(V^\#) \qquad S_{int} \in S(V^\#)[[\hbar]]$$

Zwiebach Markl

$$S_{int} = \sum_{n,G} \frac{\hbar^G}{n!} \underbrace{f_n^G(a_I)}_{\text{graded sym.}} \phi^I$$

solutions QME $\Leftrightarrow$ Loop homotopy algebras

$G = 0$ L_∞^c

Ex. 2 $\quad Ass^c \qquad S_{int}\ \sum_m f_m(a_I)\,\phi^I \qquad A_\infty^c$

$\underbrace{\quad}_{\text{cyclic sym.}}$

Stasheff

Zwiebach

Ex. 3 $\quad Mod(Ass^c) \qquad G = 2g+b-1$

$$S_{int} = \sum_{g,b,\vec{m}} \frac{\hbar^G}{b!\,m_1!\dots m_b!}\, f^{g,b}_{a_{I_1}\dots a_{I_b}}\, \phi^{I_1}\dots\phi^{I_b}$$

Doubek, Münster, B.J.

cyclic

Quantum A_∞^c

Ex. 4 $\quad$ Quantum open-closed $\qquad 1+3$ combination

Zwiebach $\qquad G = 2g+b+\frac{n_c}{2}-1$

$G=0$ classical open-closed

Stasheff-Kajiura

Ex. 5 $\quad$ type II

quantum (super) loop homotopy algebra

Münster, B.J.

Product?

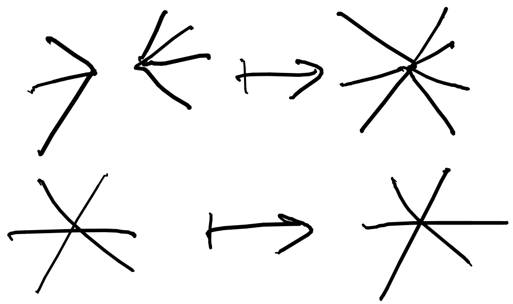
Doubek, Peksová
Pulmann, B.J.

Modular operads with connected sum #

deg 0

$\#\quad P(C,G)\otimes P(C',G') \longrightarrow P(C\sqcup C', G+G'+1)$

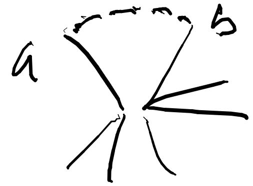
$\#\quad P(C,G) \longrightarrow P(C,G+2)$



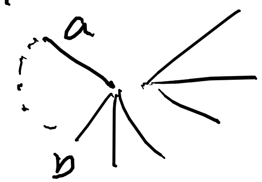
#- Σ-equivariant, associative
+ obvious compatibility conditions with ${}_a\circ_b$ and ξ_{ab}

e.g.

$\xi_{ab}\# = \#\,{}_a\circ_b$



$\#\,\xi_{ab} = \xi_{ab}\,\#$



$\vdots$

similarly for odd modular operads

Ex. 1 $\quad Mod(Com^c) \qquad (G = 2g + \frac{n}{2} - 1)$

$(n, g) \# (n', g') \longrightarrow (n+n', g+g')$ i.e. connected sum of 2 surfaces

$\#(n, g) \rightarrow (n, g+1)$ i.e. adding handle

Ex. 2 $\quad Mod(Ass^c)$ – the same interpretation as above

Ex. 3 $\quad End\, V \qquad \mathcal{E}_V(n, G) = Hom(V^{\otimes n}, \mathbb{C})$

$f \# g$ is induced by $\otimes$

$\# f = f \qquad f \in Hom(V^{\otimes n}, \mathbb{C})$ – doesn't depend on G

Algebra over $\mathcal{F}P$ gets a product $\cdot$ induced by $\#_P$ and $\#_{\mathcal{E}V}$

<u>Prop</u> $(P \otimes_{\mathbb{Z}} \mathcal{E}V, \Delta, \cdot)$ is a BV-algebra

Algebras over $\mathcal{F}P \Longleftrightarrow \Delta e^S = 0$

Remark $\Delta(f \cdot g) = \Delta f \cdot g \pm f \Delta g \pm \{f, g\}$ $\{\cdot, \cdot\}$ the same as before

Minimal model (S-matrix)

- direct application of Homological Pert. Lemma

$$V = H \oplus \operatorname{Im} Q \oplus C$$

physical — trivial — non-physical

decomposition compatible with ω $\Rightarrow$ $\omega_H = \omega|_H$

$\Delta_H = \Delta|_H$

$$h \circlearrowright (V, Q) \underset{i}{\overset{p}{\rightleftarrows}} (H, 0)$$

$$hQ + Qh = ip - 1 \quad + \ldots$$

h, Q, i, p extend to fields $\mathcal{F}(V)$ e.g. $Q = \{S_{free}, \cdot\}$

HPL $\quad h' \circlearrowright (\mathcal{F}(V), Q+\Delta) \underset{i'}{\overset{p'}{\rightleftarrows}} (\mathcal{F}(H), \Delta_H)$

- $$e^{W} := p(1-\Delta h)^{-1} e^{S_{int}}$$

$$\Delta_H e^{W} = 0$$

i.e. W defines a loop homotopy algebra on H

Theorem $$e^{W} = \int_L e^{S_{int}} \quad \left(= p\, e^{\text{propagator}} e^{S_{int}}\right)$$

$L \subset \mathrm{Im}\, Q + C \quad \hat{=}$ put trivial fields to + integrate over non-physical

propagator $= [\partial, \tilde{h}] \qquad \tilde{h} = \#_{c+t}\, h$

i.e. Minimal model $\Longleftrightarrow$ effective action

$\Delta e^{W} = 0 \qquad$ Ward id.

C. Albert
P. Mnev
A. Cattaneo
O. Gwilliam
⋮

- Works for an algebra over Feynman transform of a general operad with connected sum.

Doubek
Pelcsová
Palmer
B. J.

- HPL works also without connected sum (product)

perturbation $Q \to Q + \Delta + \{S_{int}, \cdot\}$

gives on H $\Delta_H + \{W, \cdot\}_H$

- recursion relation for S-matrix (generalisation of B.-G.)

Marcelli
Sämann
Wolf
B. J.

IBL_∞ (IBA_∞, ...) world

$$(\Delta + \{S, \cdot\})^2 = 0 \iff \text{BV QME}$$

More generally D – degree one differential operator on $\mathcal{F}(V)$

homological

$$D^2 = 0$$

$$\Updownarrow$$

Algebras over cobar construction over properads

- IBL_∞ point of view
  - nice interpretation of open-closed quantum SFT

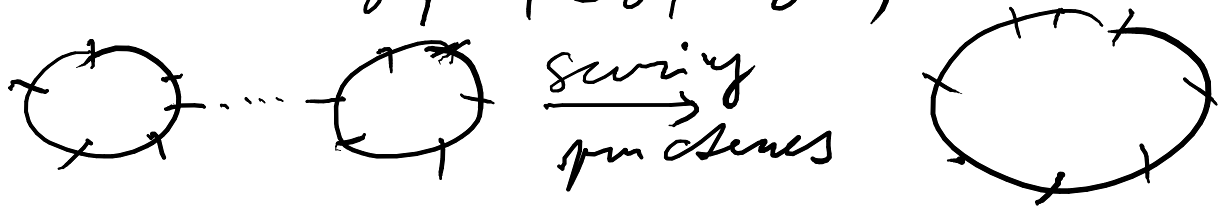
closed SFT $\qquad IBL_\infty - \mathcal{L}_c$

$(V_c, \Delta_c + \{S_c, \cdot\})$

open SFT $\qquad IBL - \mathcal{L}_o$

(cyclic cochains on V_o, $[,]_o$, δ_o)

bracket



cobracket

- open-closed SFT $\mathcal{L}_c + \mathcal{L}_{oc}$ - all but closed vertices

$\mathcal{L}_{oc}$ is an IBL_∞ morphism from $\mathcal{L}_c$ to $\mathcal{L}_o$ Münster Sachs

Zwiebach's construction of SFT - operadic interpretation

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{F}\mathcal{P} \arrow[r, "\text{decomposition}", "\text{of moduli space (geom. vertices)}"'] \arrow[dr, "\text{SFT action (alg. vertices)}"'] & \text{chains on a moduli space} \arrow[d, "\text{CFT}"] \\
& \operatorname{End} V
\end{tikzcd}$$