S-Matrix Uniqueness from Soft Theorems

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May 17, 2018

Based on work with Nima Arkani-Hamed + Jaroslav Trnka (Dec. 2016)
Fundamental limit to accuracy in QG due to black holes
Locality and unitarity break down; cannot be fundamental in QG
Lagrangian (non-manifestly deterministic) crucial and natural for Classical Mechanics (deterministic) $\rightarrow$ Quantum Mechanics (non-deterministic)
A non-manifestly local and unitarity S-matrix: a Lagrangian for the 21st century?
Main result: S-matrix is fully fixed by gauge invariance or soft theorems (including some higher order corrections): locality and unitarity emerge automatically; soft behavior contains surprisingly amount of information
Basic principles of scattering amplitudes
Locality and Unitarity

- **Locality**: singularities have a form $1/(\sum_i p_i)^2$, and can be associated to propagators of tree graphs

\[
\frac{1}{(p_1 + p_2)^2(p_1 + p_2 + p_3)^2}
\]

- **Unitarity**: when any propagator goes on-shell the amplitude must factorize into two lower point amplitudes

\[
(P)^2 A_n(1, 2\ldots n) \rightarrow A_L(1\ldots P) \times A_R(\neg P\ldots n)
\]
Basic principles of scattering amplitudes
Gauge invariance

- Amplitude must vanish when some $e_i \rightarrow p_i$
- We need gauge invariance to make Lorentz invariance, locality and unitarity manifest
- Non-trivial that the amplitude (in Feynman diagram form) is gauge invariant (needs momentum conservation, and cancellations between diagrams)
Some special scalar theories must vanish when one scalar becomes soft, $p_i = z p_i$, with $z \to 0$.

In some sense the Adler zero is like gauge invariance for scalar theories (and similarly non-trivial to see).

How fast the amplitude vanishes depends on the theory:

- Non-linear sigma model $\sim O(z)$
- Dirac-Born-Infeld $\sim O(z^2)$
- Special Galileon $\sim O(z^3)$
When a particle is taken soft, by sending $p_{n+1} = zq$, $z \to 0$, the amplitude factorizes as:

$$A_{n+1} \to \left( \frac{1}{z}S_0 + z^0S_1 + \ldots \right)A_n$$

Double soft theorems (especially for scalars)
Consider a general (ordered) local function at four points, with mass dimension matching the expected amplitude:

\[ B_4(p^2) = a_1 \frac{e_1 \cdot e_2 \cdot e_3 \cdot p_1 \cdot e_4 \cdot p_2}{p_1 \cdot p_2} + a_2 \frac{e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot p_2 \cdot p_3}{p_1 \cdot p_2} + 60 \text{ terms} \]

Impose gauge invariance, solve linear system in the \( a_i \)'s

Unique solution which matches the amplitude! Locality and Unitarity follow automatically

In general, the local ansatz will have a form

\[ B_{n}^{YM}(p^{n-2}) = \sum_{i} \frac{N_i(p^{n-2})}{P_i} \]

Locality assumption can be relaxed

Proof by induction via a soft expansion

Also works for gravity
Consider general local ansatz, fake cubic structure

\[ B_{6}^{\text{nlsym}}(p^8) = a_1 \frac{N(p^8)}{(p_1 + p_2)^2(p_1 + p_2 + p_3)^2(p_5 + p_6)^2} + \ldots \]

- Take soft limits \( p_i = z p_i, \ z \to 0 \), demand \( O(z) \) scaling
- Again a unique solution follows: the NLSM amplitude
- Proof via double soft expansion
- Crucial: no solution for lower mass dimension: \( B_{n}^{\text{nlsym}}(p^k) \) with \( [O(z)]^n \): 
  \( k < n + 2 \) no solution; \( k = n + 2 \) unique solution
- Also works for DBI, Special Galileon
Motivation for soft theorems

When doing the formal soft limit expansion proof, one begins to wonder why are there no higher order theorems? Where is the info hidden?

Higher order info is in fact present in different soft expansions

This can be used to fully constrain amplitudes, now including higher corrections (up to $F^4$ corrections for YM):

$$B_{n+1} \rightarrow (S_0 + S_1)A_n \Rightarrow B_{n+1} = A_{n+1}$$
If a term evades both $O(1/z)$ leading and $O(z^0)$ sub-leading orders, then it must go like $O(z)$ in all particles.

But this is exactly the NLSM constraint: we have shown that there is a unique object that vanishes in all soft limits, $A^{\text{nlsm}}_n(p^{n+2})$.

But YM ansatz has lower mass dimension is $B_n(p^{n-2})$ (ignoring polarization vectors) so nothing in YM ansatz can escape soft theorems.

Therefore YM amplitude is completely fixed by imposing Soft Theorems in some number of particles:

$$B_{n+1} \rightarrow (S_0 + S_1)A_n \Rightarrow B_{n+1} = A_{n+1}$$
Compare the 6 point YM amplitude with the bound $B_n^{YM}(p^{n-2})$ vs $B_n^{nlsm}(p^{n+2})$

What if we increase the mass dimension to match the bound?

Add two powers of momenta, still not possible to form a NLSM amplitude

Therefore, if we impose the soft theorem at this higher mass dimension:

$$B_{n+1} \rightarrow (S_0 + S_1)A_n^{F^3}$$

We get $B_{n+1} = A_{n+1}^{F^3}$!
Now increase by four powers, so NLSM is allowed, impose soft theorem:

\[ B_{n+1}(p^{n+3}) \rightarrow (S_0 + S_1)A_n^{F^4}(p^{n+2}) \]

For even \# we get

\[ B_{n+1} = \text{[something satisfying soft theorems]} + (e.e)^3 A_{n+1}^{nlsm} \]

For odd \# we get \( B_{n+1} = A_n^{F^4} \) (all possible five solutions: one corresponding to \((F^3)^2\), and four to \(F^4\))
Uniqueness
Soft operators and “soft” gauge invariance

- Soft theorems contain lots of info through the lower point amplitude, so maybe this is not so surprising. Can we get away with less?
- Instead of full soft theorem, only require:

\[ B_{n+1} \rightarrow (S_0 + S_1)B_n \]

- Amplitude is still unique solution (and still true for higher corrections)
- Crucially this even fixes the low point amplitude, so all the information is contained in the soft operator
- If we got this far, how about using even less info?
- Just impose gauge invariance up to sub-leading order in the soft particle. Still unique solution!
- Conclusion: soft particles carry enough information to fully constrain the amplitude
Are soft theorems independent?

Impose just leading order soft theorem

For odd $\#$, it is enough to fix the amplitude: subleading theorem doesn’t contain any new information
Uniqueness

Other theories

- This all works for GR, NLSM, DBI, even (broken) conformal dilaton theories
- GR and dilaton bound given by DBI
- GR satisfies up to $O(z^1)$ soft theorems - only DBI has $O(z^2)$ behavior
- NLSM and DBI bound given by Galileon
Easiest (ie. dumbest) way to generate amplitudes: write down ansatz, impose gauge invariance/Adler zero/soft operators

Expedites checks of various formulas

For example CHY is manifestly gauge invariant, so only need to check pole structure

It proves the BCJ double copy:

\[ \text{YM} = \sum_i \frac{c_i n_i}{s_i} \rightarrow \sum_i \frac{n_i n_i}{s_i} = \text{GR} \]

YM is gauge invariant on the support of the \( c_i \) satisfying Jacobi. If the \( n_i \) also satisfy Jacobi, then the double-copy is gauge invariant, so by uniqueness it must be the GR amplitude
Conclusion

Future questions

- There is also uniqueness from BCFW scaling - BCFW shifts in general D seem know something about Soft Theorems. Possible equivalence between BCFW scaling and soft behavior?
- Soft particles carry all amplitude information? Different perspective on BH information?
- Interesting that unitarity and locality can be derived from these abstract properties - is there some better reason for this (general inverse soft factor method)?
- Do there exist forms of the amplitude which manifest eg. correct soft behavior?
- Loops, strings?
Constructability and BCFW scaling

- Constructability means amplitudes can be built recursively, typically via a BCFW, or "on-shell" recursion.
- The recursion involves a deformation $[i, j]$ which schematically sends $p_i \to p_i + zq$ and $p_j \to p_j - zq$.
- The recursion can be used if the theory is local, unitarity, and vanishes for large $z$.

$$A_n(1, 2, \ldots, n) = \sum_i \frac{A_{i+1}(\hat{1}, \ldots, i, p)A_{n-i+1}(-p, i + 1, \ldots, \hat{n})}{(p_1 + \ldots, p_i)^2}$$

- Proven in many ways that YM amplitudes scale as $1/z$ for adjacent shifts, $1/z^2$ for non-adjacent shifts, and gravity amplitudes scale as $1/z^2$.
- Scaling at large $z$ considered mostly a (surprising) technicality, but I’ll argue it can be considered a defining property of YM, GR.
Consider the following $[i,j]$ BCFW shift:

\[
\begin{align*}
  e_i &\to \hat{e}_i & p_i &\to p_i + z\hat{e}_i \\
  e_j &\to \hat{e}_j + zp_i \frac{\hat{e}_i.e_j}{p_i.p_j} & p_j &\to p_j - z\hat{e}_i
\end{align*}
\]

where $\hat{e}_i = e_i - p_i \frac{e_i.p_j}{p_i.p_j}$.

Claim: There are unique objects which have the usual BCFW scaling under this shift (1/$z$ for adjacent, 1/$z^2$ for non-adjacent or permutation invariant functions)

Need uniqueness from Soft Theorems to prove that these objects are the amplitudes (check matching at leading and subleading order)

Strongest possible claim: simple polynomial fixed to amplitude numerators by BCFW scaling

BCFW scaling implies locality, unitarity, gauge invariance
Not completely trivial relation between the action of a BCFW shift and sub-leading operator:

\[ S_i = e^{[\mu \ q^\nu]} J^\mu_* = e^{[\mu \ q^\nu]} \frac{1}{q.p_i} \left( e^{\mu}_i \frac{\partial}{\partial e^\nu_i} + p^\mu_i \frac{\partial}{\partial p^\nu_i} \right) \]  \hspace{2cm} (1)

\[ K_i \equiv e^\mu q^\nu J^\mu_* \]  \hspace{2cm} (2)

Consider some polynomial \( f \), which doesn’t depend on \( e, q \)

Easy to see:

\[ \text{BCFW}_{[q,i]}[f] = f + z K_i[f] + \mathcal{O}(z^2) \]  \hspace{2cm} (3)

Schematically explains why \( S_0 A_n + S_1 A_n \approx \mathcal{O}(z^{-1}) \)

Surprisingly close connection between soft operator and BCFW shift...both completely fix amplitude...something deeper going on?