S-Matrix Uniqueness from Soft Theorems

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Motivation
Why scattering amplitudes

- Fundamental limit to accuracy in QG due to black holes
- Locality and unitarity break down; cannot be fundamental in QG
- Lagrangian (non-manifestly deterministic) crucial and natural for Classical Mechanics (deterministic) $\rightarrow$ Quantum Mechanics (non-deterministic)
- A non-manifestly local and unitarity S-matrix: a Lagrangian for the 21st century?
- Main result: S-matrix is fully fixed by gauge invariance or soft theorems (including some higher order corrections): locality and unitarity emerge automatically; soft behavior contains surprisingly amount of information

Basic principles of scattering amplitudes
Locality and Unitarity

- Locality: singularities have a form $1/(\sum_i p_i)^2$, and can be associated to propagators of tree graphs

$$\frac{1}{(p_1+p_2)^2(p_1+p_2+p_3)^2}$$

- Unitarity: when any propagator goes on-shell the amplitude must factorize into two lower point amplitudes

$$(P)^2 A_n(1,2...n) \to A_L(1...P) \times A_R(-P...n)$$

Basic principles of scattering amplitudes
Gauge invariance

- Amplitude must vanish when some $e_i \to p_i$
- We need gauge invariance to make Lorentz invariance, locality and unitarity manifest
- Non-trivial that the amplitude (in Feynman diagram form) is gauge invariant (needs momentum conservation, and cancellations between diagrams)

Basic principles of scattering amplitudes
Adler zero

- Some special scalar theories must vanish when one scalar becomes soft, $p_i = zp_i$, with $z \to 0$
- In some sense the Adler zero is like gauge invariance for scalar theories (and similarly non-trivial to see)
- How fast the amplitude vanishes depends on the theory:

$$\text{Non-linear sigma model} \sim \mathcal{O}(z)$$
$$\text{Dirac-Born-Infeld} \sim \mathcal{O}(z^2)$$
$$\text{Special Galileon} \sim \mathcal{O}(z^3)$$

- When a particle is taken soft, by sending $p_{n+1} = zq$, $z \to 0$, the amplitude factorizes as:

$$A_{n+1} \to (\frac{1}{z} S_0 + z^0 S_1 + \ldots) A_n$$

- Double soft theorems (especially for scalars)

Uniqueness
Gauge invariance

- Consider a general (ordered) local function at four points, with mass dimension matching the expected amplitude:

$$B_4(p^2) = a_1 \frac{e_1.e_2\ e_3.p_1\ e_4.p_2}{p_1.p_2} + a_2 \frac{e_1.e_2\ e_3.e_4\ p_2.p_3}{p_1.p_2} + 60 \text{ terms}$$

- Impose gauge invariance, solve linear system in the a_i's
- Unique solution which matches the amplitude! Locality and Unitarity follow automatically
- In general, the local ansatz will have a form

$$B_n^{\text{YM}}(p^{n-2}) = \sum_i \frac{N_i(p^{n-2})}{P_i}$$

- Locality assumption can be relaxed
- Proof by induction via a soft expansion
- Also works for gravity

- Consider general local ansatz, fake cubic structure

$$B_6^{\text{nlsm}}(p^8) = a_1 \frac{N(p^8)}{(p_1+p_2)^2(p_1+p_2+p_3)^2(p_5+p_6)^2} + \ldots$$

- Take soft limits $p_i = zp_i$, $z \to 0$, demand $\mathcal{O}(z)$ scaling
- Again a unique solution follows: the NLSM amplitude
- Proof via double soft expansion
- Crucial: no solution for lower mass dimension: $B_n^{\text{nlsm}}(p^k)$ with $[\mathcal{O}(z)]^n$: $k < n+2$ no solution; $k = n+2$ unique solution
- Also works for DBI, Special Galileon

Motivation for soft theorems

- When doing the formal soft limit expansion proof, one begins to wonder why are there no higher order theorems? Where is the info hidden?
- Higher order info is in fact present in different soft expansions
- This can be used to fully constrain amplitudes, now including higher corrections (up to F^4 corrections for YM):

$$B_{n+1} \to (S_0 + S_1)A_n \Rightarrow B_{n+1} = A_{n+1}$$

- If a term evades both $\mathcal{O}(1/z)$ leading and $\mathcal{O}(z^0)$ sub-leading orders, then it must go like $\mathcal{O}(z)$ in all particles
- But this is exactly the NLSM constraint: we have shown that there is a unique object that vanishes in all soft limits, $A_n^{\text{nlsm}}(p^{n+2})$
- But YM ansatz has lower mass dimension is $B_n(p^{n-2})$ (ignoring polarization vectors) so nothing in YM ansatz can escape soft theorems
- Therefore YM amplitude is completely fixed by imposing Soft Theorems in some number of particles

$$B_{n+1} \to (S_0 + S_1)A_n \Rightarrow B_{n+1} = A_{n+1}$$

- Compare the 6 point YM amplitude with the bound $B_n^{\mathrm{YM}}(p^{n-2})$ vs $B_n^{\mathrm{nlsm}}(p^{n+2})$
- What if we increase the mass dimension to match the bound?
- Add two powers of momenta, still not possible to form a NLSM amplitude
- Therefore, if we impose the soft theorem at this higher mass dimension:

$$B_{n+1} \to (S_0 + S_1)A_n^{F^3}$$

- We get $B_{n+1} = A_{n+1}^{F^3}$!

- Now increase by four powers, so NLSM is allowed, impose soft theorem:

$$B_{n+1}(p^{n+3}) \to (S_0 + S_1)A_n^{F^4}(p^{n+2})$$

- For even # we get

$$B_{n+1} = [\text{something satisfying soft theorems}] + (e.e)^3 A_{n+1}^{nlsm}$$

- For odd # we get $B_{n+1} = A_{n+1}^{F^4}$ (all possible five solutions: one corresponding to $(F^3)^2$, and four to F^4)

Uniqueness
Soft operators and "soft" gauge invariance

- Soft theorems contain lots of info through the lower point amplitude, so maybe this is not so surprising. Can we get away with less?
- Instead of full soft theorem, only require:

$$B_{n+1} \to (S_0 + S_1)B_n$$

- Amplitude is still unique solution (and still true for higher corrections)
- Crucially this even fixes the low point amplitude, so all the information is contained in the soft operator
- If we got this far, how about using even less info?
- Just impose gauge invariance up to sub-leading order in the soft particle. Still unique solution!
- Conclusion: soft particles carry enough information to fully constrain the amplitude

- Are soft theorems independent?
- Impose just leading order soft theorem
- For odd #, it is enough to fix the amplitude: subleading theorem doesn't contain any new information

- This all works for GR, NLSM, DBI, even (broken) conformal dilaton theories
- GR and dilaton bound given by DBI
- GR satisfies up to $\mathcal{O}(z^1)$ soft theorems - only DBI has $\mathcal{O}(z^2)$ behavior
- NLSM and DBI bound given by Galileon

Conclusion
Some practical implications

- Easiest (ie. dumbest) way to generate amplitudes: write down ansatz, impose gauge invariance/Adler zero/soft operators
- Expedites checks of various formulas
- For example CHY is manifestly gauge invariant, so only need to check pole structure
- It proves the BCJ double copy:

$$\mathrm{YM} = \sum_i \frac{c_i n_i}{s_i} \to \sum_i \frac{n_i n_i}{s_i} = \mathrm{GR}$$

YM is gauge invariant on the support of the c_i satisfying Jacobi. If the n_i also satisfy Jacobi, then the double-copy is gauge invariant, so by uniqueness it must be the GR amplitude

- There is also uniqueness from BCFW scaling - BCFW shifts in general D seem know something about Soft Theorems. Possible equivalence between BCFW scaling and soft behavior?
- Soft particles carry all amplitude information? Different perspective on BH information?
- Interesting that unitarity and locality can be derived from these abstract properties - is there some better reason for this (general inverse soft factor method) ?
- Do there exist forms of the amplitude which manifest eg. correct soft behavior?
- Loops, strings?

Bonus
Constructability and BCFW scaling

- Constructability means amplitudes can be built recursively, typically via a BCFW, or "on-shell" recursion
- The recursion involves a deformation $[i, j\rangle$ which schematically sends $p_i \to p_i + zq$ and $p_j \to p_j - zq$
- The recursion can be used if the theory is local, unitarity, and vanishes for large z

$$A_n(1, 2, \ldots, n) = \sum_i \frac{A_{i+1}(\hat{1}, \ldots, i, p)A_{n-i+1}(-p, i+1, \ldots, \hat{n})}{(p_1 + \ldots, p_i)^2}$$

- Proven in many ways that YM amplitudes scale as $1/z$ for adjacent shifts, $1/z^2$ for non-adjacent shifts, and gravity amplitudes scale as $1/z^2$
- Scaling at large z considered mostly a (surprising) technicality, but I'll argue it can be considered a defining property of YM, GR

Bonus
Uniqueness from BCFW scaling

- Consider the following $[i, j\rangle$ BCFW shift:

$$
\begin{aligned}
e_i &\to \hat{e}_i & \qquad p_i &\to p_i + z\hat{e}_i \\
e_j &\to \hat{e}_j + z p_i \frac{\hat{e}_i . e_j}{p_i . p_j} & \qquad p_j &\to p_j - z\hat{e}_i
\end{aligned}
$$

  where $\hat{e}_i = e_i - p_i \frac{e_i . p_j}{p_i . p_j}$.
- Claim: There are unique objects which have the usual BCFW scaling under this shift ($1/z$ for adjacent, $1/z^2$ for non-adjacent or permutation invariant functions)
- Need uniqueness from Soft Theorems to prove that these objects are the amplitudes (check matching at leading and subleading order)
- Strongest possible claim: simple polynomial fixed to amplitude numerators by BCFW scaling
- BCFW scaling implies locality, unitarity, gauge invariance

Bonus
Relation between BCFW and Soft theorems?

- Not completely trivial relation between the action of a BCFW shift and sub-leading operator:

$$S_i = e^{[\mu} q^{\nu]} J_i^{\mu\nu} = e^{[\mu} q^{\nu]} \frac{1}{q.p_i} \left(e_i^\mu \frac{\partial}{\partial e_i^\nu} + p_i^\mu \frac{\partial}{\partial p_i^\nu} \right) \tag{1}$$

$$K_i \equiv e^\mu q^\nu J_i^{\mu\nu} \tag{2}$$

- Consider some polynomial f, which doesn't depend on e, q
- Easy to see:

$$\mathrm{BCFW}_{[q,i\rangle}[f] = f + z\, K_i[f] + \mathcal{O}(z^2) \tag{3}$$

- Schematically explains why $S_0 A_n + S_1 A_n \approx \mathcal{O}(z^{-1})$
- Surprisingly close connection between soft operator and BCFW shift...both completely fix amplitude...something deeper going on?