From null geodesic to gravitational scattering

An alternative route from BMS to soft theorems via ambitwistor strings

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Solvay BMS, infrared, etc., 16 May, 2018

Ambitwistor-strings based on arxiv:1406.1462 with Yvonne Geyer and Arthur Lipstein following on from work by Adamo, Casali & Skinner and work with Skinner. Builds on CHY. Conformal scattering, w/ Jean-Philippe Nicolas,

Null geodesics and scattering

$(M^d, g) =$ space-time with Lorentzian metric g which

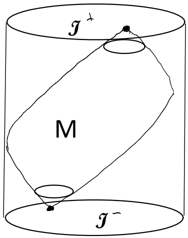
- is Globally hyperbolic, asymptotically flat/de Sitter
- has conformal compactification $\widetilde{M} = M \cup \mathscr{I}^+ \cup \mathscr{I}^-$,
- null geodesics end on $\mathscr{I}^-$ in past and $\mathscr{I}^+$ in future.

Scattering thru M gives symplectic maps for

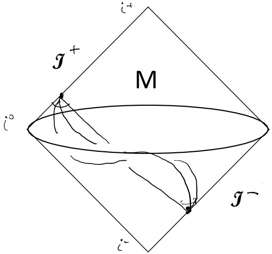
- null geodesics $T^*\mathscr{I}^- \to T^*\mathscr{I}^+$
- Gravitational field data on $\mathscr{I}^-$ to data at $\mathscr{I}^+$.

Ambitwistor strings gives formulae:

$$\text{Gravity S-matrix} = \langle \text{Hamiltonians for light ray scattering} \rangle_{string}$$



Asymptotically de Sitter



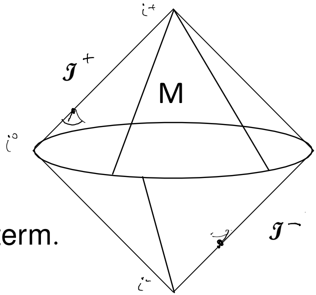
Asymptotically Flat

Contents

- Conformal scattering and S-matrix,
- Ambitwistor strings,
- CHY formulae for S-matrix,
- Ambitwistor strings at $\mathscr{I}$,
- Asymptotic symmetries and soft theorems.

Conformal scattering and gravity tree S-matrix

- Pose asymptotic data $g_{\rm in}$: resp. $\pm$ frequency at $\mathscr{I}^{\pm}$.
- Solve for g on M s.t. $\pm$ freq. parts at $\mathscr{I}^{\pm}$ agree with $g_{\rm in}$.
- S-matrix is functional of $g_{\rm in}$



$$\mathcal{S}[g_{in}] = S_{EH}[g] := \frac{1}{\kappa^2}\int_M R\, d\, vol + \text{bdy term}.$$

- generating fn $g^{\mp}|_{\mathscr{I}^{\pm}} = \partial\mathcal{S}/\partial g^{\pm}_{\rm in}$.

Object: Compute/study $\mathcal{S}$ as a functional on the gravitational phase space $\mathcal{P}$ of data at $\mathscr{I}$.

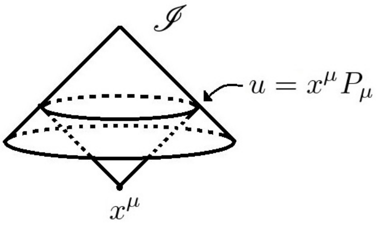
Geometry of $\mathscr{I}$

In 4-dims:

- $\mathscr{I} = S^2 \times \mathbb{R} =$ light cone of origin under inversion.
- Coordinatise $S^{d-2} = \{p^\mu | p^2 = 0, p_\mu \sim \alpha p^\mu\}, \mu = 0, \dots, 3$ and $\mathbb{R}$-factor by u with scaling

$$\mathscr{I} = \{(u, p_\mu) | p^2 = 0\} / \{(u, p_\mu) \sim (\alpha u, \alpha p_\mu)\}.$$

  So vertex is $u = \infty$ (inverted from p^μ / u).
- Flat space lightcone of $x^\mu \in M$, intersects $\mathscr{I}$ at:



Framework extends simply to all dimensions.

Gravitational phase space $\mathcal{P}$

[Ashtekar 1981]

Choose coords with $p_\mu = (1, \zeta, \bar{\zeta}, \zeta\bar{\zeta})$ and space-time coordinates $(R, u, \zeta, \bar{\zeta})$ with $R = \frac{1}{r} = 0$ on $\mathscr{I}$ so that metric is

$$R^2 \mathrm{d}s^2 = \mathrm{d}u\mathrm{d}R + \mathrm{d}\zeta\mathrm{d}\bar{\zeta} + R(\sigma\mathrm{d}\zeta^2 + c.c.) + O(R^2)\,.$$

- gravity data is shear $\sigma(u, \zeta, \bar{\zeta})\mathrm{d}\zeta^2$.
- Finite mass $\Leftrightarrow \partial_u\sigma \in L^2_{\mathscr{I}}$.
- Supertranslations $u \to u + f(\zeta, \bar{\zeta})$ act by:

$$\sigma \to \sigma + \partial_\zeta^2 f\,.$$

- Symplectic structure $\Omega(\sigma, \sigma') = \int_{\mathscr{I}} (\bar{\sigma}\partial_u\sigma' + c.c.)\mathrm{d}u\mathrm{d}^2\zeta\,.$

Scattering $\sigma_{\mathscr{I}^-} \to \sigma_{\mathscr{I}^+}$ determined by $\delta\mathcal{S}[\sigma]/\delta\sigma$.

The perturbative S-matrix

Usually evaluate S-matrix perturbatively

- Pose data $\sigma_{\rm in} = \sum_{i=1}^{n} \epsilon_i \sigma_i|_{\mathscr{I}}$, and solve for g on M.
- For Einstein

$$S_{EG}[g] = \frac{1}{\kappa^2}\int_M R\, d\, vol + \int_{\partial M} K\, d\, vol_{\partial M},$$

and (tree) S-matrix is

$$\mathcal{M}(g_1, \dots, g_n) = \text{ Coeff of } \prod_i \epsilon_i \text{ in } S_{EG}[g]$$

Use Fourier modes for g_j: $\quad g_{j\mu\nu} = \xi_{j\mu}\xi_{j\nu}\mathrm{e}^{ik_j\cdot x}$.

- momentum k_j, $k_j^2 = 0$.
- polarization data satisfies

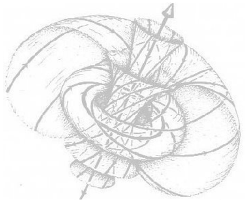
$$k\cdot\xi = 0\,, \qquad \xi \sim \xi + \alpha k\,.$$

For n-particle scattering $\mathcal{M}(1, \dots, n) = \mathcal{M}(k_1, \xi_1, \dots, k_n, \xi_n)$.

Compute by Feynman diagrams or ambitwistor-strings.

Ambitwistors

Ambitwistor spaces: spaces of complex null geodesics $\mathbb{A}$.



- Extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.
- Yang-Mills Witten and Isenberg, et. al. 1978, 1985.
- Conformal and Einstein gravity LeBrun [1983,1991] Baston & M. [1987] .

Ambitwistor Strings (strings at $\alpha' = 0$ for field theories):

- Twistor-string for $N = 4$ Yang-Mills [Witten, Roiban, Spradlin, Volovich, 2003/4].
- $N = 8$ supergravity [Cachazo, Geyer, Skinner, M., 2012], [Skinner, 2013]
- Tree S-Matrices in all dimensions for gravity, YM etc. [CHY]
- From strings in ambitwistor space [M. & Skinner 2013] $\rightsquigarrow$ vast generalization of original twistor-string; many theories & dimensions (i.e., Einstein-YM, DBI, BI, NLSM).
- Gives worldsheet version of soft theorems $\leftrightarrow$ BMS without Ward identities at $\mathscr{I}$ (in contrast with Strominger et. al.).

[Geyer, Lipstein, M 1406.1462.]

Geometry of ambitwistor space

Complexify: real d-diml space-time $(M_{\mathbb{R}}, g_{\mathbb{R}}) \rightsquigarrow (M, g)$.

- $\mathbb{A} :=$ space of scaled complex null geodesics.
- For $(P_\mu, X^\nu) \in T^*M$ let $D_0 := P \cdot \nabla =$ geodesic spray.

$$\mathbb{A} = T^*M|_{P^2=0}/\{D_0\}$$

- D_0 has Hamiltonian P^2 wrt symplectic form $\omega = dP_\mu \wedge dx^\mu$.
- Symplectic potential $\theta = P_\mu dx^\mu$, $\omega = d\theta$, descend to $\mathbb{A}$.

Study with double fibration

$$\begin{array}{ccc} & T^*M|_{P^2=0} & \\ q \swarrow D_0 & & \searrow \\ \mathbb{A} & & M\,. \end{array}$$

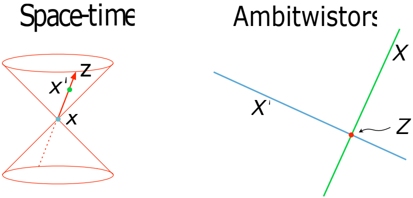
LeBrun correspondence

Projectivise: $P\mathbb{A} :=$ space of *unscaled* complex light rays.

- On $P\mathbb{A}$, θ defines a holomorphic contact structure.

Theorem (LeBrun 1983)

The complex structure on $P\mathbb{A}$ determines M and conformal metric g. The correspondence is stable under arbitrary deformations of the complex structure of $P\mathbb{A}$ that preserve θ.



Linearized LeBrun correspondence

θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:

$$\text{Deformations of complex structure } \leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)\,.$$

Key example: On flat space-time, set $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_{\mu\nu}$ then

$$\delta\theta = \bar\delta(k\cdot p)\mathrm{e}^{ik\cdot q/w}\epsilon_{\mu\nu}p^\mu p^\nu\ ,$$

where $\bar\delta(z) = \bar\partial \frac{1}{2\pi i z}$.

- Dolbeault form of Penrose's scattering Hamiltonian for null geodesics from $\mathscr{I}^-$ to $\mathscr{I}^+$,

$$\delta\theta = \bar\partial h, \qquad h = \mathrm{e}^{ik\cdot q/w}\frac{\epsilon_{\mu\nu}p^\mu p^\nu}{k\cdot p} = \int_\gamma \delta g_{\mu\nu}p^\mu p^\nu ds$$

- $h =$ gravitational Wilson-line (Hamilton-Jacobi fn for null geodesic scattering $T*\mathscr{I}^- \to T^*\mathscr{I}^+$).
- Support on $k\cdot p = 0 \Rightarrow$ the *scattering equations*.

Ambitwistor strings

Take Riemann surface $\Sigma \ni \sigma$, want holomorphic maps $\Sigma \to \mathbb{A}$.

- Let $X^\mu(\sigma) : \Sigma \to M$, $P_\mu \in T^*_X M \otimes \Omega^{1,0}_\Sigma$.

$$S = \int P_\mu \bar{\partial} X^\mu - e\, P_\mu P^\mu / 2\,.$$

  with $e \in \Omega^{0,1} \otimes T$, $T = T^{1,0}\Sigma$.
- $e \rightsquigarrow P^2 = 0$,
- gauge: $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$.

Solutions mod gauge are holomorphic maps to

$$\mathbb{A} = T^*M|_{P^2=0} / \{\text{gauge}\}.$$

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Solutions mod gauge are holomorphic maps to

$$\mathbb{A} = T^* M|_{P^2=0} / \{\text{gauge}\}.$$

For gravity must add type II worldsheet susy $S_{\Psi_1} + S_{\Psi_2}$ where

$$S_\Psi = \int_\Sigma \Psi_\mu \bar{\partial} \Psi^\mu + \chi P \cdot \Psi\,.$$

Gravity Vertex operators and CHY

- NS sector of type II SUGRA $\delta g_{\mu\nu} + \delta B_{\mu\nu} = \epsilon_{1\mu}\epsilon_{2\nu}\mathrm{e}^{ik\cdot x}$ gives

$$\begin{aligned} V_i &= \int_\Sigma \delta\theta \\ &= \int_\Sigma \mathrm{e}^{ik\cdot q/w}\,\bar\delta(k\cdot p)\prod_{r=1}^{2}\epsilon_{r\mu}(p^\mu + \Psi_r^\mu k\cdot\Psi_r)\,, \\ &= \oint \frac{\mathrm{e}^{ik\cdot q/w}}{k\cdot p}\prod_{r=1}^{2}\epsilon_{r\mu}(p^\mu + \Psi_r^\mu k\cdot\Psi_r)\,. \end{aligned}$$

contour around $k\cdot p = 0$ last equality from $\bar\delta(k\cdot p) = \bar\partial\frac{1}{k\cdot p}$.

- There are also 2 fixed vertex operators U_i needed.
- **Key result:** Correlator gives CHY gravity formula:

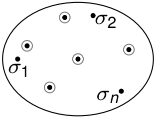
$$\langle U_1 U_2 V_3 \dots V_n\rangle_{string} = \int \frac{Pf_l Pf_r}{vol\ SL_2\times\mathbb{C}^3}\prod_{i=1}^{n}\bar\delta(k_i\cdot P(\sigma_i))d\sigma_i$$

The scattering equations

Take n null momenta $k_i \in \mathbb{R}^d$, $i = 1, \dots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{CP}^1 \to \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}\,, \qquad \sigma, \sigma_i \in \mathbb{CP}^1$$



- Solve for $\sigma_i \in \mathbb{CP}^1$ with the n *scattering equations* [Fairlie 1972]

$$\mathrm{Res}_{\sigma_i}\left(P^2\right) = k_i \cdot P(\sigma_i) = \sum_{j=1}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

$$\Rightarrow \quad P^2 = 0 \ \forall \sigma.$$

- For Mobius invariance $\Rightarrow P \in \mathbb{C}^d \otimes \Omega^{1,0}\mathbb{CP}^1$
- There are $(n-3)!$ solutions.

Arise in large α' strings [Gross-Mende 1988] & twistor-strings [Roiban, Spradlin, Volovich, Witten 2004].

Amplitude formulae for massless theories.

Proposition (Cachazo, He, Yuan 2013,2014)

Tree-level massless amplitudes in d-dims are integrals/sums

$$\mathcal{M}_n = \delta^d\left(\sum_i k_i\right)\int_{(\mathbb{CP}^1)^n}\frac{\mathcal{I}^l\mathcal{I}^r\prod_i\bar\delta(k_i\cdot P(\sigma_i))}{\text{Vol SL}_2\times\mathbb{C}^3}$$

where $\mathcal{I}^{l/r}=\mathcal{I}^{l/r}(\epsilon_i^{l/r},k_i,\sigma_i)$ depend on the theory.

- polarizations ϵ_i^l for spin 1, $\epsilon_i^l\otimes\epsilon_i^r$ for spin-2 ($k_i\cdot\epsilon_i=0\dots$).
- Introduce skew $2n\times 2n$ matrices $M=\begin{pmatrix}A & C\\ -C^t & B\end{pmatrix}$,

  $$A_{ij}=\frac{k_i\cdot k_j}{\sigma_i-\sigma_j},,\quad B_{ij}=\frac{\epsilon_i\cdot\epsilon_j}{\sigma_i-\sigma_j},\quad C_{ij}=\frac{k_i\cdot\epsilon_j}{\sigma_i-\sigma_j},\qquad \text{for } i\neq j$$

  and $A_{ii}=B_{ii}=0$, $C_{ii}=\epsilon_i\cdot P(\sigma_i)$.
- For Yang-Mills, $\mathcal{I}^l=Pf'(M)$, $\mathcal{I}^r=\prod_i\frac{1}{\sigma_i-\sigma_{i-1}}$.
- For Gravity $\mathcal{I}^l=Pf'(M^l)$, $\mathcal{I}^r=Pf'(M^r)$.

Ambitwistors at $\mathscr{I}$: $\mathbb{A} = T^*\mathscr{I}$

- $\mathscr{I}$ = light cone coordinatised with $\frac{p^\mu}{u}$ null (vertex $u = \infty$)

$$\mathscr{I} = \{(u, p_\mu) | p^2 = 0\} / \{(u, p_\mu) \sim (\alpha u, \alpha p_\mu)\}.$$

- Let (w, q^μ) be momenta for (u, p_μ), so

$$\theta = w \, \mathrm{d}u - q^\mu \, \mathrm{d}p_\mu \, .$$

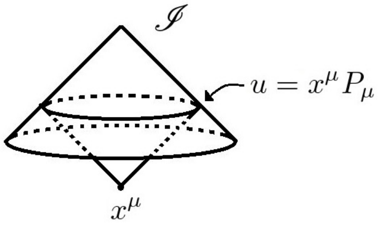
- Scaling α acts by $(w, q^\mu) \sim (w/\alpha, q^\mu/\alpha)$, generated by

$$wu - p \cdot q.$$

- Hamiltonian quotient $wu - p \cdot q = 0 \rightsquigarrow$ original model by

$$u = p_\mu x^\mu \, , \qquad x^\mu = q^\mu / w \, , \quad P_\mu = w \, p_\mu .$$

gives incidence:



Ambitwistor strings at $\mathscr{I}$

- For gravity must include SUSY with Fermionic Ψ_r^μ, $r = 1, 2$

$$\theta = w\,\mathrm{d}u - q^\mu\,\mathrm{d}p_\mu + \Psi_r^\mu \mathrm{d}\Psi_{r\mu}\,.$$

  with supersymmetries generated by $w\,p\cdot\Psi_r$.

- Worldsheet action gauges constraints $p^2 = wu - p\cdot q = 0$:

$$S = \int_\Sigma w\,\bar\partial u - q^\mu\,\bar\partial p_\mu + \Psi_r^\mu \bar\partial \Psi_{r\mu} + ep^2 + a(wu - p\cdot q) + \chi_r wp\cdot\Psi_r\,.$$

- Usual string proposal

$$\mathcal{M}(g_1, \dots, g_n) = \int D[u, q, p, \dots]\, V_1 \dots V_n\, \mathrm{e}^{iS} =: \langle V_1 \dots V_n \rangle$$

  where V_i = *vertex operators* $= \delta\theta \leftrightarrow \delta g_{\mu\nu}$ etc..

Action of BMS

For us BMS group are diffeos of $\mathscr{I}$ consisting of

- Supertranslations

$$(u, p_\mu) \to (u + f(p), p_\mu)\,.$$

  reducing to translations if $f(p) = a^\mu p_\mu$.
- Rotations generated by $r_{\mu\nu} = r_{[\mu\nu]}$, i.e., $\delta p_\mu = -r_\mu^\nu p_\nu$.
- Super-rotations when $r_{\mu\nu}(p)$ depends on p of weight 0.

On $\mathbb{A}$ these are respectively generated by Hamiltonians

$$H_f = w\, f(p)\,, \qquad H_r = r_{\mu\nu}(p^\mu q^\nu + w\Psi_r^\mu \Psi_r^\nu)$$

- Superrotations that are not rotations are never symmetries.
- Supertranslations are only symmetries in dimension $d = 4$.

Diffeomorphisms of $\mathbb{A}$ and worldsheet charges

Hamiltonian diffeos of $\mathbb{A}$ generated by Hamiltonian $h(u, p_\mu, w, q^\mu)$ represented by worldsheet charges

$$Q_h = \oint h$$

- Vertex operators are examples that arise from

$$h = \frac{\mathrm{e}^{ik\cdot q/w}}{k\cdot p}\prod_{r=1}^{2}\epsilon_{r\mu}(p^\mu + \Psi_r^\mu k\cdot\Psi_r)$$

  diffeo that corresponds to $\mathscr{I}^- \to \mathscr{I}^+$ scattering.
- Supertranslations and (super)-rotations

$$Q_f = \oint w\, f(p)\,, \qquad Q_r = \oint r_{\mu\nu}(p) J^{\mu\nu}$$

  where

$$J^{\mu\nu} = (p^{[\mu}q^{\nu]} + w\Psi_r^\mu\Psi_r^\nu)$$

  is the angular momentum operator incorporating spin.

Subleading soft theorems

Theorem (Low, Weinberg, Cachazo, Strominger, ...)

Let $k_{n+1} = s \to 0$ *and expand in s. We have*

$$\mathcal{M}(1,\ldots,n+1) \to (S_0 + S_1 + \ldots)\mathcal{M}(1,\ldots,n)\,,$$

where

$$S_0 = \sum_{a=1}^{n} \frac{(\epsilon \cdot k_a)^2}{s \cdot k_a}, \qquad S_1 = \frac{\epsilon_{\mu\nu} k_a^{\mu} s_\lambda J_a^{\lambda\nu}}{s \cdot k_a}, \qquad S_2 = \frac{\epsilon_{\mu\nu}(s_\lambda J_a^{\lambda\mu})(s_\rho J_a^{\rho\nu})}{s \cdot k_a}$$

$J^{\mu\nu}$ *is the angular momentum operator incorporating spin.*

Subleading soft theorems from the worldsheet

- In ambitwistor string, expand the vertex op $V_{n+1} = V_s$ in s:

$$V_s = \oint \frac{w\, \mathrm{e}^{is\cdot q/w}}{s\cdot p} \prod_{r=1}^{2} \epsilon_{r\mu}(p^\mu + i\Psi_r^\mu \Psi_r \cdot s) = V_s^0 + V_s^1 + V_s^2 + V_s^3 + \ldots .$$

where $V_s^0 = \oint w \frac{(\epsilon\cdot p)^2}{s\cdot p}$,

$$V_s^1 = \oint \frac{\epsilon \cdot p}{s\cdot p} \epsilon^\mu s^\nu \left(p_{[\mu}\, q_{\nu]} + w \sum_{r=1}^{2} \Psi_{r\mu}\Psi_{r\nu} \right) = \oint \frac{\epsilon\cdot p}{s\cdot p} \epsilon^\mu s^\nu J_{\mu\nu} .$$

- V_s^0 generates super-translation and V_s^1 super-rotations

$$f(p) = \frac{(\epsilon\cdot p)^2}{s\cdot p}\, , \qquad r^{\mu\nu}(p) = \frac{\epsilon\cdot o \epsilon^{[\mu} s^{\nu]}}{s\cdot p}$$

- Proof: $\{$residue at $s\cdot p = 0\} = \{\sum$ residues at $\sigma_s - \sigma_i = 0\}$.
- **Slogan:** Soft graviton = supertrans. + superrotation + ...
- At higher order $\rightsquigarrow$ diffeo of $T^*\mathscr{I}$ not lifted from $\mathscr{I}$.

Summary

- Gravity vertex operators $=$ generators of diffeos of $\mathbb{A}$ for null geodesic scattering through space-time.
- These are essentially gravitational 'Wilson lines'.
- On-shell condition is quantum worldsheet consistency.
- In soft limit: gravity vertex op $\rightarrow$ gen. BMS generator.
- Our BMS is *not* symmetry; always singular even in 4d.
- Distinction from Strominger et. al.:
  - they use Ashtekar Fock space & divide gravitons into hard and soft parts, obtain soft theorem as Ward identity.
  - We re-interpret worldsheet gravity vertex op as *singular* diffeo; these are *not* symmetries, no Ward identity.
- Appropriate 2d CFT for Strominger argument is that of ambitwistor string!
- Manifests BCJ double copy, YM vs GR.

Outlook

- Similar story at horizons?
- definitions on curved backgrounds, cf [Adamo, Casali & Skinner 2014] and recent work [Adamo, Casali, M. & Nekovar 2017-8].

The end

Thank You!