Gravitational Wave Memory and an Electromagnetic Analog

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"Infrared Physics: Asymptotic and BMS symmetry, soft theorems, memory, information paradox ... "

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Overview

- Solving Einstein's Equations
- Spacetimes and Energy
- Gravitational Radiation with Memory
  - Various Situations
- An Electromagnetic Analog of Memory

Photo: Courtesy of ETH-Bibliothek Zürich.

A. Einstein with mathematician A. Hurwitz and his daugther L. Hurwitz.

Spacetimes

Definition

Spacetimes (M, g), where M a 4-dimensional manifold with Lorentzian metric g solving Einstein's equations:

$$\mathbf{G}_{\mu\nu} := \mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{g}_{\mu\nu}\,\mathbf{R} = \mathbf{2}\,\mathbf{T}_{\mu\nu}\;,$$

where
$\mathbf{G}_{\mu\nu}$ is the Einstein tensor,
$\mathbf{R}_{\mu\nu}$ is the Ricci curvature tensor,
$\mathbf{R}$ the scalar curvature tensor,
$\mathbf{g}$ the metric tensor and
$\mathbf{T}_{\mu\nu}$ denotes the energy-momentum tensor.

Asymptotically Flat versus Cosmological Spacetimes

In the cosmological case, we add to the original Einstein equations the term containing $\mathbf{\Lambda}$, the positive cosmological constant:

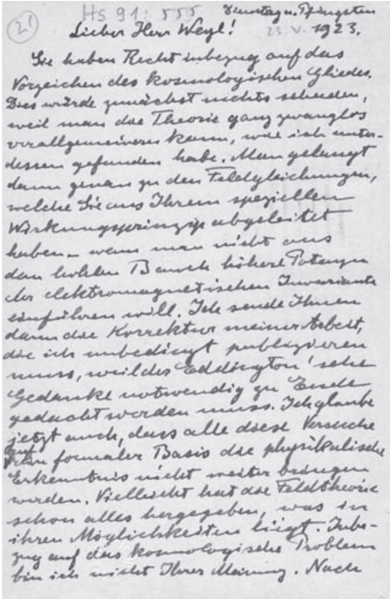
$$\mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}}\, \mathbf{g}_{\mu\nu}\, \mathbf{R} + \mathbf{\Lambda} \mathbf{g}_{\mu\nu} = \mathbf{8}\pi\, \mathbf{T}_{\mu\nu}\ , \tag{1}$$

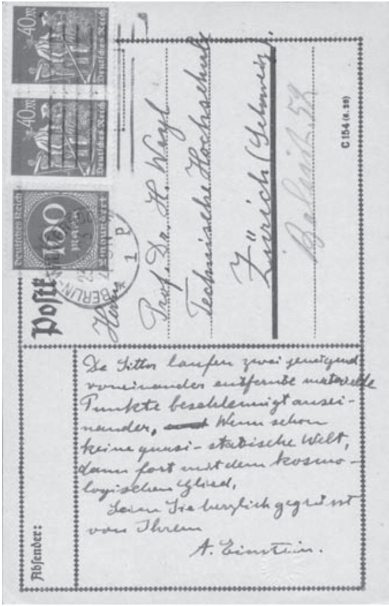
Asymptotically Flat Spacetimes: Fall-off (in particular of metric and curvature components) towards Minkowski spacetime at infinity. Natural definition of "null infinity" $\Rightarrow$ understand gravitational radiation.

These are solutions of the original Einstein equations with asymptotically flat initial data.

Cosmological Spacetimes: Solutions of the cosmological Einstein equations (1). "Null infinity is spacelike". $\Rightarrow$ no "natural" way to discuss radiation.

Postcard from Albert Einstein to Hermann Weyl, 1923





Courtesy of ETH-Bibliothek Zürich

Fig. 9.5 Lemaître and Einstein. Georges Lemaître (1894–1966) and Albert Einstein (1879–1955), photographed around 1933. (Archives Lemaître, Université Catholique, Louvain.)

A Few Early Highlights

- 1915: Albert Einstein formulates the field equations of GR.
- 1916: Karl Schwarzschild derives first solution to Einstein equations.
- 1917: Einstein introduces the cosmological term into the Einstein equations. First cosmological solutions by Albert Einstein and Willem de Sitter.
- 1919: Arthur Eddington's expedition confirms light bending.
- 1922: Alexander Friedmann derives dynamical cosmological solutions.
- 1927: Georges Lemaître derives further dynamical solutions and combines them with Vesto Slipher's observations of redshifts in galaxies and with Edwin Hubble's distance measurements. Lemaître derives that the Universe is expanding. He computes the linear velocity-distance relationship $v = H \cdot d$.
- 1929: Edwin Hubble confirms this relation.

Energy in GR

At a point the gravitational field can be transformed away.

How to define energy?

For certain systems, we can define quasi-local energies, or energy of a spacelike hypersurface or energy of a null hypersurface.

Albert Einstein formulated an energy-momentum theorem for his closed universe , most of his colleagues did not agree with its formulation:

"`While the general relativity theory was approved by most theoretical physicists and mathematicians, almost all colleagues object to my formulation of the energy-momentum theorem.`"

(Albert Einstein, 1918. "Der Energiesatz in der allgemeinen Relativitätstheorie." SAW, 448-459. Translated from German.)

From the Bianchi identity

$$\nabla_{[\alpha} R_{\beta\gamma]\delta\varepsilon} \ = \ 0$$

we compute the twice contracted Bianchi identity

$$\nabla^j G_{ij} = 0 \quad .$$

This implies that

$$\nabla^j T_{ij} = 0.$$

However...

Noether Theorems

Noether: Within the setting of a Lagrangian theory to each continuous group of transformations leaving the Lagrangian invariant there corresponds a quantity which is conserved.

Photo: public domain.

GR - “A Special Universe”

In GR $\Rightarrow$ General spacetimes do not have any symmetries.

However... certain things can be done.

Let’s first see how Einstein, Hilbert and Weyl struggled towards finding “energy components of the gravitational field”.

Hermann Weyl on Conserved Quantities

Hermann Weyl wrote the following about invariant conserved quantities. From the book: "Raum-Zeit-Materie" ("Space-Time-Matter"), Hermann Weyl, 1921, Springer-Verlag. Translated from German:

Nevertheless it seems to be physically meaningless to introduce the t_i^k (Einstein pseudo-tensor based on a specific Lagrangian) as energy components of the gravitational field; for, these quantities are *neither a tensor nor are they symmetric.* In fact by choosing an appropriate coordinate system all the t_i^k can be made to vanish at any given point; ... Although the differential relations (referring to a divergence of the Einstein pseudo-tensor being zero) are without a physical meaning, nevertheless by *integrating them over an isolated system* one gets invariant conserved quantities.

Energies Control the Curvature

Use Bel-Robinson Tensor to do Energy Estimates

Bel-Robinson tensor:
Associate to a Weyl field a tensorial quadratic form:

- a 4-covariant tensorfield
- being fully symmetric and trace-free.

$$Q_{\alpha\beta\gamma\delta} \ = \ \frac{1}{2}\left(W_{\alpha\rho\gamma\sigma}\, W_{\beta\ \ \delta}^{\ \rho\ \ \sigma} \ + \ {}^*W_{\alpha\rho\gamma\sigma}\ \ {}^*W_{\beta\ \ \delta}^{\ \rho\ \ \sigma}\right) .$$

It satisfies the following positivity condition:

$$Q\ (X_1,\ X_2,\ X_3,\ X_4)\ \geqslant\ 0$$

X_1, X_2, X_3, X_4 future-directed timelike vectors. For W satisfying the Bianchi equations:

$$D^{\alpha}\ Q_{\alpha\beta\gamma\delta}\ =\ 0\ .$$

`Notation: Hodge duals` ${}^*W$ `and` W^* `defined as`

$$\begin{aligned} {}^*W_{\alpha\beta\gamma\delta} &= \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta} \\ W^*_{\alpha\beta\gamma\delta} &= \frac{1}{2}W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta} \end{aligned}$$

Current

Contract $Q_{\alpha\beta\gamma\delta}$ with three future-directed causal vectorfields X, Y, Z

to obtain a current

$$J_\alpha := Q_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta \tag{2}$$

Apply the divergence theorem
on a bounded domain $\Omega \subset M$ in the spacetime M.

Curvature Flux:

If $\partial\Omega$ contains a portion of a null hypersurface C with affine tangent null vectorfield L, then the corresponding boundary term is the curvature flux through C and is given by

$$\mathcal{F} = \int_C J_\alpha L^\alpha d\mu_{g_C} \tag{3}$$

with $d\mu_{g_C}$ being the canonical volume form on C associated to L.

Solving Einstein Equations

Want: Answers to problems in physics and astrophysics, insights into mathematical structures of spacetimes.

Have to: Give initial data describing situations from astrophysics and physics and then solve the Einstein equations.

In order to understand the dynamics of the gravitational field, stability properties, and to get information on the asymptotics

$\Rightarrow$ We have to solve the initial value problem for the Einstein equations for various classes of spacetimes.

$\Rightarrow$ Investigate their structures $\Rightarrow$ to derive information about gravitational radiation and memory.

Caution: Even if a spacetime is a solution to the Einstein equations or to Einstein equations coupled to other fields or matter, then in general, one does not know their local and global behavior nor their asymptotics. All kinds of behavior could change the asymptotics. Only the full mathematical investigation of these spacetimes gives an answer.

“The Beginnings”

“Pioneers” include (more names to be added...)

D. Hilbert, H. Weyl, A.S. Eddington, W. de Sitter, A. Friedmann, G. Lemaître, T. de Donder, C. Lanczos, G. Darmois, A. Lichnerowicz, J. Leray, K. Stellmacher, K. Friedrichs, K. Schwarzschild, Y. Choquet-Bruhat. On the analysis side, important progress that influenced GR came with the works by H. Lewy, J. Hadamard, J. Schauder and S. Sobolev among many others. - Photos of Hermann Weyl, Yvonne Choquet-Bruhat.

Global Solutions - Stability of Minkowski Space

Semiglobal Result: [H. Friedrich (1986)]

Global Result by S. Klainerman and D. Christodoulou, 1991, proving the global nonlinear stability of Minkowski spacetime.

Theorem [D. Christodoulou and S. Klainerman for EV (1991)]

Every asymptotically flat initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime at infinity along any geodesic.

Generalizations of the Christodoulou-Klainerman Result:

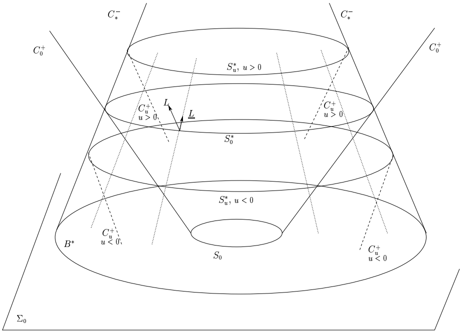
[N. Zipser for EM (2000)] Generalization for Einstein-Maxwell case.

[L. Bieri for EV (2007)] Generalization in the Einstein-vacuum case obtaining borderline estimates for decay.

All the above: geometric-analytic proofs, exact solutions.

Long list of other results and partial results. Works by many authors: Including but not complete: Y. Choquet-Bruhat, H. Friedrich, R. Geroch, S. Hawking, H. Lindblad, F. Nicolò, R. Penrose, I. Rodnianski, and more.

Spacetime



Foliation by a time function t
$\Rightarrow$ spacelike, complete Riemannian hypersurfaces H_t.

Foliation by a function u
$\Rightarrow$ null hypersurfaces C_u.

$$S_{t,u} \ = \ H_t \ \cap \ C_u$$

Future Null Infinity

Null Infinity

Future null infinity $\mathcal{I}^+$ *is defined to be the endpoints of all future-directed null geodesics along which* $r \to \infty$. *It has the topology of* $\mathbb{R} \times \mathbb{S}^2$ *with the function* u *taking values in* $\mathbb{R}$.

Thus a null hypersurface C_u intersects $\mathcal{I}^+$ at infinity in a 2-sphere $S_{\infty,u}$.

Black Hole Region

We define the black hole region of an asymptotically flat spacetime (M, g) *to be the set of points* $B \subset M$ *not in the past of future null infinity* $\mathcal{I}^+$. *We write* $B = M \backslash J^-(\mathcal{I}^+)$.

Remark: *The causal past* $J^-(p)$ of a point $p \in M$ is defined to be the set of all points $q \in M$ for which there exists a past-directed causal curve initiating at p and ending at q.

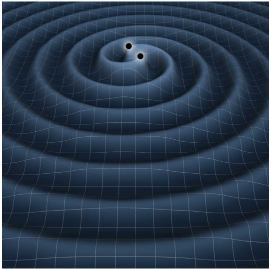
Gravitational Waves - Energy Radiated

Fluctuation of curvature of the spacetime

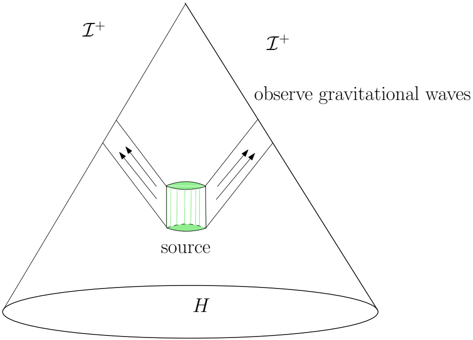
propagating as a wave.

Gravitational waves:

Localized disturbances in the geometry propagate at the speed of light, along outgoing null hypersurfaces.



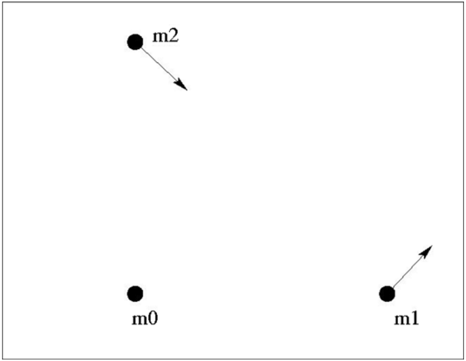
Picture: Courtesy of NASA.



Memory Effect of Gravitational Waves

Gravitational waves traveling from their source to our experiment. Three test masses in a plane as follows. The test masses will experience

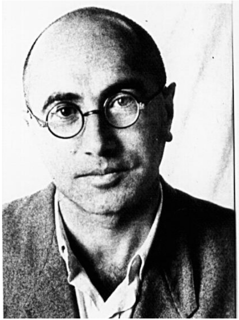
1. Instantaneous displacements (while the wave packet is traveling through)
2. Permanent displacements (cumulative, stays after wave packet passed). This is called the memory effect of gravitational waves. Two types of this memory.



Memory - Continued - Isolated Systems

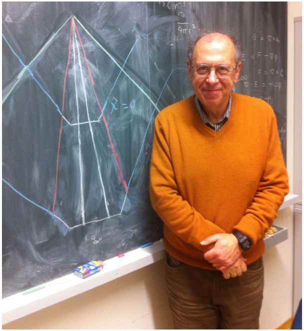
Ordinary (formerly called "linear") effect
=> was known for a long time in the slow motion limit [Ya.B. Zel'dovich, A.G. Polnarev 1974]

Null (formerly called "nonlinear") effect
=> was found by [D. Christodoulou 1991].



Ya. B. Zel'dovich

A. G. Polnarev



D. Christodoulou

Early Works on Memory

T. Damour, L. Blanchet, V. B. Braginsky, L. P. Grishchuk, C. M. Will , A. G. Wiseman, K. S. Thorne, J. Frauendiener.

Other Related Early Works:
[A. Ashtekar and various co-authors (1970s and 1980s)] Studies of asymptotic symmetries in GR and infrared problems in quantum field theory.

Memory - Continued - Isolated Systems

Contribution from electro-magnetic field to null effect
=> was found by [L. Bieri, P. Chen, S.-T. Yau 2010 and 2011].

Contribution from neutrino radiation to null effect
=> was found by [L. Bieri, D. Garfinkle 2012 and 2013].

For the first time outside of GR, for pure Maxwell equations:

We find an electromagnetic analog of gravitational wave memory.
[L. Bieri, D. Garfinkle 2013]

$\Rightarrow$ charged test masses observe a residual kick.

Other theories: In recent years, A. Strominger relates memory effect, soft theorem and asymptotic symmetry to each other. Many papers by many authors.

Recent works on memory include Wald, Tolish, Favata, Flanagan, Nichols, Strominger, Winicour, Loutrel, Yunes, Hawking, Perry, Zhiboedov, Pasterski and more.

Ordinary and Null

- Ordinary Memory: Sourced by $\triangle P$, that is the change in the radial component of the electric part of the Weyl tensor.

- Null Memory: Sourced by F:

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} \left(| \, \Xi \, |^2 + C \cdot T^* \right) du$$

where Ξ denotes the asymptotic shear introduced before, C is a positive constant, T^* the limit of the outgoing null component of the stress-energy tensor (which exists and is positive for many physical spacetimes), and $\frac{\mathbf{F}}{\mathbf{4\pi}}$ is the total energy radiated to infinity in a given direction per unit solid angle.

Concentrating on the Null Memory

Null Memory

$\Rightarrow$ Christodoulou Memory: F contains only part including the shear Ξ. (D. Christodoulou)

$\Rightarrow$ Positive contribution from T^*

- for electromagnetic fields (Einstein-Maxwell equations) (L. Bieri, P. Chen, S.-T. Yau),
- for neutrino radiation (Einstein-null-fluid) (L. Bieri, D. Garfinkle)
- for a “fairly general” stress-energy tensor with decay r^{-2} in the outgoing null direction (L. Bieri, D. Garfinkle).

A paper by P. Lasky, E. Thrane, Y. Levin, J. Blackman and Y. Chen suggests a method for detecting gravitational wave memory with aLIGO by stacking events.

For a situation where the geodesics are not too far away from each other,

$\Rightarrow$ one can replace the geodesic equation for γ_1 and γ_2 by the Jacobi equation (geodesic deviation from γ_0).

$$\frac{d^2x^k}{dt^2} \;=\; -\;R_{kTlT}\;x^l \tag{4}$$

with

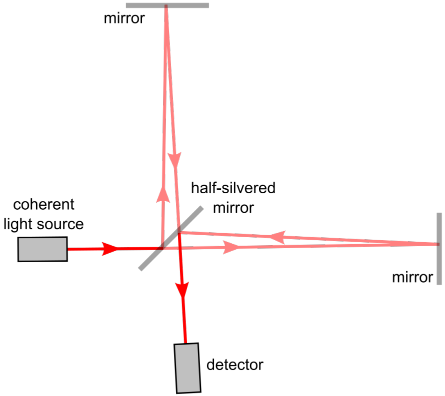
$$R_{kTlT} \;=\; R\;(E_k, T, E_l, T)$$

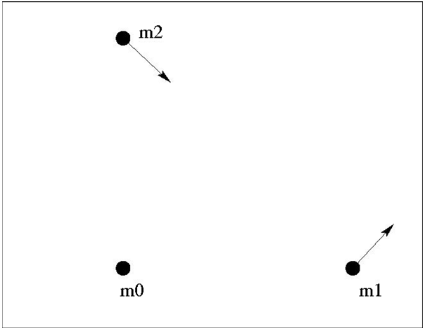
where $k, l = 1, 2, 3$.

Information about the curvature and null structures required!

$\Rightarrow$ Analyze the spacetimes!

From Mathematical Theory to Physics and Observation





Memory - Permanent Displacement

Asymptotically Flat Spacetimes

The permanent displacement of test masses is related to the difference $(\Sigma^+ \; - \; \Sigma^-)$ in the asymptotic shears, which themselves depend on the radiated energy in a nonlinear way.

$$\triangle x \; = \; - \, (\frac{d_0}{r}) \, (\Sigma^+ \; - \; \Sigma^-) \; . \tag{5}$$

There are the following contributions to the permanent displacement $\triangle x$:

The ordinary memory is sourced by $\triangle P$, that is the change in the radial component of the electric part of the Weyl tensor. The null memory is sourced by F, the energy radiated to infinity (including shear and component of energy-momentum tensor).

Electromagnetic (EM) Memory

We find an electromagnetic analog of gravitational wave memory.
[L. Bieri, D. Garfinkle 2013]

$\Rightarrow$ charged test masses observe a residual kick.

Detection

Detectors of electromagnetic radiation $\Rightarrow$ absorb energy from the wave.

• Flux of energy in the wave: goes as r^{-2}.

• Sensitivity of the detector falls off like r^{-2}.

Detectors of gravitational waves $\Rightarrow$ sensitivity falls off like r^{-1}.

• Gravitational wave detector works not by measuring power absorbed from the wave but rather by following the motion induced in the detector by the wave.

What permanent changes occur?

- Gravitational: Permanent displacement.
- Electromagnetic: Residual velocity (kick).

Electromagnetic (EM) Memory

Motion of a charge in the presence of an electromagnetic wave.

Charged test masses $\Rightarrow$ Measure residual velocity (= kick).

For a charge q with mass m the equation of motion is

$$m\frac{d^2\vec{x}}{dt^2} = q\vec{E} \tag{6}$$

It follows that once the wave has passed the charge has received a kick given by

$$\Delta\vec{v} = \frac{q}{m}\int_{-\infty}^{\infty}\vec{E}dt \tag{7}$$

Ordinary Memory

Slow motion limit and far from the source:

$$\vec{E} = \frac{1}{r} P \left[\frac{d^2 \vec{p}}{dt^2} \right] \tag{8}$$

where $\vec{p}$ is the dipole moment of the source and $P[]$ denotes "projected orthogonal to the radial direction." It then follows that the kick is given by

$$\Delta \vec{v} = \frac{q}{mr} P \left[\frac{d}{dt} \vec{p}(t = \infty) - \frac{d}{dt} \vec{p}(t = -\infty) \right] \tag{9}$$

Consider only systems which at large positive and negative times consist of widely separated charges each moving at constant velocity. Then it is

$$\frac{d}{dt} \vec{p} = \sum_k q_k \vec{v}_k \tag{10}$$

where the sum is over all objects, where the kth object has charge q_k and velocity $\vec{v}_k$. $\Rightarrow$ Then the kick is given by

$$\Delta \vec{v} = \frac{q}{mr} P \left[\sum_k q_k \vec{v}_k (t = \infty) - \sum_k q_k \vec{v}_k (t = -\infty) \right] \tag{11}$$

Null Memory

General case $\Rightarrow$ kick still given by equation (7) from before:

$$\Delta\vec{v} = \frac{q}{m}\int_{-\infty}^{\infty}\vec{E}dt \tag{12}$$

But: cannot use far field slow motion expression (eqn. (8)) or the electric field.

$\Rightarrow$ Instead: analyze the general behavior of the electromagnetic field far from the source.

Recall: electric and magnetic fields E_a and B_a satisfy Maxwell's equations:

$$\partial_a E^a = 4\pi\rho \tag{13}$$

$$\partial_a B^a = 0 \tag{14}$$

$$\partial_t B_a + \varepsilon_{abc}\partial^b E^c = 0 \tag{15}$$

$$\partial_t E_a - \varepsilon_{abc}\partial^b B^c = -4\pi j_a \tag{16}$$

where ρ is the charge density and j_a is the current density.

Null Memory Continued

Maxwell's equations in spherical coordinates.

Spherical coordinate indices: write r for radial direction and capital latin letters for two-sphere direction.

$$\partial_r E_r + 2r^{-1} E_r + r^{-2} D_A E^A = 4\pi\rho \tag{17}$$

$$\partial_r B_r + 2r^{-1} B_r + r^{-2} D_A B^A = 0 \tag{18}$$

$$\partial_t B_r + r^{-2} \varepsilon^{AB} D_A E_B = 0 \tag{19}$$

$$\partial_t E_r - r^{-2} \varepsilon^{AB} D_A B_B = -4\pi j_r \tag{20}$$

$$\partial_t B_A + \varepsilon_A{}^B (D_B E_r - \partial_r E_B) = 0 \tag{21}$$

$$\partial_t E_A - \varepsilon_A{}^B (D_B B_r - \partial_r B_B) = -4\pi j_A \tag{22}$$

Here D_A and ε_{AB} are respectively the derivative operator and volume element of the unit two-sphere, and all indicies are raised and lowered with the unit two-sphere metric.

Null Memory Continued

Expand all quantities in inverse powers of r with expansion coefficients that are functions of retarded time $u = t - r$ and the angular coordinates.

For an electromagnetic field that is smooth at null infinity it follows that

$$E_A = X_A + \dots \tag{23}$$

$$B_A = Y_A + \dots \tag{24}$$

$$E_r = W r^{-2} + \dots \tag{25}$$

$$B_r = Z r^{-2} + \dots \tag{26}$$

$$\rho = j_r = r^{-2} L + \dots \tag{27}$$

where $\dots$ means "terms higher order in r^{-1}" and we also assume that at large r the angular components of j_a are negligible compared to the radial component.

Null Memory Continued

Consider a field that is both charged and massless. This is the analog for electromagnetism of fields whose stress-energy gets out to null infinity.

Closer look at equation (27):

$$\rho = j_r = r^{-2}L + \ldots$$

Introduce the current density four-vector J^μ given by $J^t = \rho$ and $J^a = j^a$. We also introduce the advanced time $v = t + r$. It then follows that $J_u = -\frac{1}{2}(j_r + \rho)$ and $J_v = \frac{1}{2}(j_r - \rho)$. Thus the behavior given in (27) is equivalent to

$$\begin{aligned} J_u &= -r^{-2}L + \ldots && (28) \\ J_v &= O(r^{-3}) && (29) \\ J_A &= O(r^{-3}) && (30) \end{aligned}$$

Null Memory Continued

Define the quantity S_A by

$$S_A = \int_{-\infty}^{\infty} X_A du \tag{31}$$

Then it follows that S_A satisfies the equations

$$D_A S^A = (W(\infty) - W(-\infty)) + 4\pi F \tag{32}$$
$$\varepsilon^{AB} D_A S_B = Z(-\infty) - Z(\infty) \tag{33}$$

where the quantity F is defined by

$$F = \int_{-\infty}^{\infty} L du \tag{34}$$

$\Rightarrow$ It follows that the kick points in the direction of S^A and has a magnitude of

$$\Delta v = \frac{q}{mr} |S^A| \tag{35}$$

EM Memory consists of ordinary kick and null kick .

EM Memory : Analyzed

EM Memory consists of ordinary kick and null kick .

$$\Delta v = \frac{q}{mr}|S^A| \tag{36}$$

- ordinary kick due to difference between the early and late time values of the radial component of the electric field E_r
- null kick due to charge radiated to infinity, that is F giving the amount of charge radiated to infinity per unit solid angle.

Fast Charged Particle Mimics Null Memory

Measure EM Memory

- Ordinary and null memory are distinct just as timelike particles differ from null particles.
- Just as a timelike particle with high velocity mimics a null particle $\Rightarrow$ So the ordinary memory can mimic the null memory.
- In an experiment: A very fast charged (timelike) particle $\Rightarrow$ ordinary memory mimics null memory.
- This is in principle measurable as residual velocities (i.e. kicks) of charged test masses of a detector.

Cosmology: de Sitter, FLRW and ΛCDM

Observations of 1998 of Distant Supernovae

$\Rightarrow$ Accelerating Expansion of the Universe

Most popular cosmological theories:

- ΛCDM (with cold (i.e. non-relativistic) dark matter)
- Friedmann-Lemaître-Robertson-Walker (FLRW) (with a perfect fluid)
- de Sitter (dS) (modeling early inflation period of the Universe)

Positive cosmological constant.

de Sitter Spacetime and FLRW

- (L. Bieri, D. Garinkle, S.-T. Yau)

We find in de Sitter spacetime, that there is a factor of $(1 + rH_0)$ multiplying F, the energy per unit solid angle radiated to infinity.

- For FLRW (A. Tolish, R. Wald):

For sources at the same luminosity distance, the memory effect in a spatially flat FLRW spacetime is enhanced over the Minkowski case by a factor of $(1 + z)$.

Thus, the in de Sitter and FLRW spacetimes

$\Rightarrow$ null memory is enhanced by redshift factor.

FLRW: Friedmann - Lemaître - Robertson - Walker

The FLRW metric reads

$$ds^2 = -dt^2 + a^2(t)\big(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\big)$$

Universe started as a small perturbation from FLRW.

$\Rightarrow$ by now: these perturbations have grown

$\Rightarrow$ waves propagate through highly inhomogeneous medium.

Consider gravitational waves in ΛCDM cosmology.

ΛCDM

Our inhomogeneous spacetime

$\Rightarrow$ two zones: "wave zone" and "cosmological zone".

(L. Bieri, D. Garfinkle, N. Yunes) For gravitational wave memory we find that in the wave zone the memory is "similar" to the one with Minkowski as a background, whereas in the cosmological zone the memory is given by the memory in the wave zone multiplied by a factor including the redshift and a magnification factor due to lensing.

Use the approximation that the wavelength of the waves is short compared to all other scales in the problem.

Consider a background solution of the Einstein field equations with dust and a cosmological constant: metric $\bar{g}_{ab}(x^\mu)$, dust density $\bar{\rho}(x^\mu)$ and four-velocity $\bar{u}_a(x^\mu)$ satisfying

$$\bar{R}_{ab} - \tfrac{1}{2}\bar{R}\bar{g}_{ab} + \Lambda\bar{g}_{ab} - 8\pi\bar{\rho}\,\bar{u}_a\bar{u}_b = 0 \quad . \tag{37}$$

This background represents the cosmology of our evolving universe, which we take to be FLRW on large scales, though with (possibly large) density contrasts on small scales.

Next, add perturbations.

Introduce: one-parameter family of tensor fields $\hat{g}_{ab}(x^\mu, \xi)$, $\hat{\rho}(x^\mu, \xi)$, $\hat{u}_a(x^\mu, \xi)$, and a scalar field $\phi(x^\mu)$.

These are high-frequency deformations of the background uniformly bounded in ξ.

Only restriction $\Rightarrow$ length scale of inhomogeneities large compared to the much smaller wavelength of the gravitational waves.

The full spacetime and matter content of the universe is then given by the one-parameter family of tensor fields (g_{ab}, ρ, u_a):

$$
\begin{aligned}
g_{ab} &= \bar{g}_{ab}(x^\mu) + \omega^{-2}\hat{g}_{ab}(x^\mu, \omega\phi(x^\mu))\,. && (38)\\
\rho &= \bar{\rho}(x^\mu) + \omega^{-1}\hat{\rho}(x^\mu, \omega\phi(x^\mu))\,, && (39)\\
u_a &= \bar{u}_a(x^\mu) + \omega^{-1}\hat{u}_a(x^\mu, \omega\phi(x^\mu))\,, && (40)
\end{aligned}
$$

where ω is the frequency of the perturbations. The fields (g_{ab}, ρ, u_a) represent our universe in the sense that they satisfy the Einstein-fluid equations *to the appropriate order*:

$$
R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} - 8\pi\rho\, u_a u_b = O(\omega^{-2}) \quad , \qquad (41)
$$

Equation (41) describes the waves only in the region away from their sources.

Parameter ω plays a dual role:

$\Rightarrow$ the frequency of the perturbation

$\Rightarrow$ and as an inverse amplitude.

The surfaces $\phi = \mathrm{const}$ are wavefronts, since in the large ω limit the waves vary rapidly in the direction perpendicular to them.

Remarks

- Different from usual perturbative approach in GR, where in the latter one assumes that there is a one-parameter family of metric tensor fields, where each member of the family is an exact solution of the field equations, but one only calculates that family to first order in the parameter.

- Instead in the weak progressive wave approach, the one-parameter family of metric tensor fields is not expanded only to first order in the parameter, but rather the field equations themselves, (Eq. (41)) are only satisfied to a given order in ω^{-1}.

- Our approach yields results that are gauge invariant.

Solutions to $O(\omega^{-1})$

$\Rightarrow$ Gravitational perturbations (which travel at the speed of light) and fluid perturbations (which travel at the speed of sound, in this case zero because the fluid is dust) cannot have the same wavevector. Thus a perturbation with a single wavevector must be pure gravity or pure fluid.

$\Rightarrow$ A perturbation with a single wavevector must be pure gravity or pure fluid.

Let us now consider a non-trivial gravitational perturbation.

Let $k_a = \nabla_a \phi$. Prime denotes derivative w.r.t. ξ and $\bar{\nabla}_a$ takes derivatives w.r.t. x^μ.

Since the fluid perturbation vanishes at lowest order, it follows that $R^{(1)}_{ab} = 0$. That is, even to $O(\omega^{-1})$ the field equations reduce to that of vacuum. From $R^{(1)}_{ab} = 0$ and $P'{}_a = 0$ we obtain

$$-k^b \bar{\nabla}_b \hat{g}'_{ac} - \tfrac{1}{2}(\bar{\nabla}_b k^b)\hat{g}'_{ac} + k_{(a}L_{c)} = 0\,. \tag{42}$$

$L_a = \bar{\nabla}_b \hat{g}'_a{}^b - \frac{1}{2}\bar{\nabla}_a \hat{g}'$ is pure gauge. $\Rightarrow$ Up to terms that are pure gauge, the fall-off of the gravitational wave amplitude is determined by the properties of the divergence of the null geodesic vector field k^a. This result can be stated in a manifestly gauge invariant way as follows. Taking the derivative with respect to ξ of Eq. (42) we derive

$$k^e \bar{\nabla}_e C^{(0)}_{abcd} = -\tfrac{1}{2}(\bar{\nabla}_e k^e) C^{(0)}_{abcd}\,. \tag{43}$$

Implications in Homogeneous Background Spacetimes

The background spacetimes we consider are FLRW on large scales, but on small scales the null geodesics can encounter curvature that can lead to modifications in wave propagation.

Since light rays are described by null geodesics, the effect of lensing on the brightness of a light wave is given by an equation of the same form as Eq. (43). We show that the additional effect of inhomogeneities is precisely to multiply the Weyl tensor by a *magnification factor* due to gravitational lensing.

Consider a scalar $\mathcal{A}$ that satisfies the equation

$$k^e \bar{\nabla}_e \mathcal{A} = -\frac{1}{2} \mathcal{A} \bar{\nabla}_e k^e \,. \tag{44}$$

We can then use this equation to rewrite Eq. (43) as

$$k^e \bar{\nabla}_e C^{(0)}_{abcd} - \mathcal{A}^{-1} C^{(0)}_{abcd} k^e \bar{\nabla}_e \mathcal{A} = 0 \,. \tag{45}$$

Now: Specialize to an inhomogeneous FLRW spacetime background, described by

$$ds^2 \quad = -\left(1+2\Phi\right)a^2(\tau)d\tau^2 + a^2(\tau)\left(1-2\Psi\right)\delta_{ij}dx^i dx^j\,, \qquad (46)$$

where δ_{ij} is the Kronecker delta and (Φ, Ψ) are matter inhomogeneities that in principle depend on conformal time τ (related to the time coordinate t via $dt = a(\tau)d\tau$) and the Cartesian coordinates x^i. Such a perturbed FLRW spacetime suggests similar perturbative decompositions of other quantities, such as the scalar function $\mathcal{A} = \mathcal{A}_0\left(1+\zeta\right)$, where $\mathcal{A}_0$ and ζ are independent and linearly-dependent on the matter inhomogeneities respectively. It is $\mathcal{A}_0 = 1/(ar)$.

The part of $\mathcal{A}$ that is linearly proportional to the matter inhomogeneities can be obtained by solving Eq. (44) linearized in (Φ, Ψ). This equation, in turn, depends on the solution to the null-geodesic equation in the perturbed spacetime of Eq. (46).
P. Laguna, S.L. Larson, D. Spergel, N. Yunes studied such inhomogeneities. Combining with their findings we compute

$$\begin{aligned} \zeta &= \Psi - \Psi_e \\ &+ \frac{1}{2}\int_0^{\lambda} \frac{d\lambda'}{(\lambda')^2} \int_0^{\lambda'} d\lambda''\, D_A D^A (\Phi + \Psi) \quad . \end{aligned} \tag{47}$$

where λ is the affine parameter of null geodesics in the non-expanding but inhomogeneous spacetime (Eq. (46) with $a(\tau)$ set to unity), Ψ_e is the value of this Ψ at emission, and $D_A D^A$ is the Laplacian on the unit two sphere. The first term in the above equation corresponds to the standard Sachs-Wolfe effect, while the second is a magnification due to lensing.

With $\mathcal{A}$ calculated, we then find that Eq. (43) in the spacetime of Eq. (46) simplifies to

$$k^f \bar{\nabla}_f \left[C^{(0)}_{bcde} a r \left(1 - \zeta\right) \right] = 0 \,, \tag{48}$$

where once more we have expanded in small matter inhomogeneities.

Next, we shall show that the ζ term magnifies the signal, thus magnifying the memory effect.

Cosmological Memory

The wave zone is defined through the asymptotic relation $H_0^{-1} \gg r \gg \lambda$, while the cosmological zone is defined through $r \gtrsim H_0^{-1}$, where r is the distance from the gravitational wave emitting source to a field point, λ is the gravitational wave wavelength and H_0 is the Hubble parameter today.

Now, a gravitational wave is emitted at $r_0 = 0$,

$\Rightarrow$ detected first at r_1 in the wave zone

$\Rightarrow$ and then detected again at r_2 in the cosmological zone.

Recall: For two nearby geodesics with four-velocity u^a and separation s^a acted on by a gravitational wave with Weyl tensor C_{abcd}, the geodesic deviation equation requires that

$$\ddot{s}^a = -C^a{}_{bcd} u^b s^c u^d \tag{49}$$

where an overdot denotes derivative with respect to the proper time of the geodesics.

For simplicity we assume an initial displacement orthogonal to the direction of propagation of the wave, and we consider only the memory due to energy radiated to infinity. Capital letters denote indices in this two-sphere of orthogonal directions.

Measurements in the wave zone can be related to measurements in the cosmological zone through the definition of the luminosity distance.

In Minkowski spacetime, the luminosity distance is the same as the usual r coordinate.

The luminosity distance is given by

$d_L = ra(1+z)$,

where z is the redshift and a is the scale factor at the location of the measurement.

Equation (49) implies that after the wave has passed there will be a residual change in the separation Δs^a.

Let the original separation s be in the B direction. Then the change in separation Δs in the A direction is given by

$$\Delta s = -\frac{s}{d_L} m^A{}_B \tag{50}$$

where the memory tensor $m^A{}_B$ is given by

$$m^A{}_B = \int_{-\infty}^{\infty} d\tilde{t} \int_{-\infty}^{\tilde{t}} dt \, \left(d_L C_{abcd} x^a u^b y^c u^d \right) \quad . \tag{51}$$

Here x^a and y^a are respectively unit vectors in the A and B directions.

Redshift and Gravitational Lensing

We obtain

for the memory in the cosmological zone

$$m_{AB}^{(2)} = (1 + z_2)\,(1 + \zeta_2)\,m_{AB}^{(1)} \tag{52}$$

where ζ_2 induces a magnification or a demagnification due to lensing (analogous to focusing and de-focusing) of the signal, $m_{AB}^{(1)}$ denotes the memory tensor in the wave zone, z_2 is the redshift at distance r_2 in the cosmological zone.

Further Questions

- Various systems coupled to Einstein equations.

$\Rightarrow$ How do they change the patterns of gravitational waves and memory?

- Different geometric-analytic structures.

- Analogs in other physical theories.

$\cdots$ and many more.

Thank you!