Hierarchies of asymptotic conditions and results

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1 Introduction

Because asymptotically flat solutions to Einstein's field equations arises from data on non-compact initial hypersurfaces any work on such solutions depends on assumptions about the asymptotic behaviour of the data and the resulting properties of the solutions.

A lot of work has been done on this topic. To get a complete picture, however, and to understand which choices may be too restrictive or redundant some questions still need to be answered.

In the following I shall give some overview and point out some questions. I shall only be concerned with the asymptotic structure at space-like and null infinity and ignore the evolution of black holes and the behaviour of solutions near time-like infinity.

For authors and technicalities, abundant in existence proofs but ignored here, I refer to the references given at the end of the talk. But I need to recall some ideas most of you may be familiar with.

2 Basic idea

Trautman-Bondi-Sachs-Newman-Penrose consider far fields of isolated systems.

− They introduce the idea of future null infinity $\mathcal{J}^+_*$:

$\exists$ caustic-free family of outgoing null hypersurfaces $\{u = const.\}$ ruled by future complete null geodesics with parameter r so that $r \to \infty$ defines $\mathcal{J}^+_* \sim \mathbb{R} \times \mathbb{S}^2$ (Bondi coordinates).

− They combine this with a notion of asymptotic flatness:

Bondi-Sachs: In the given coordinates the solutions approximate the Minkowski metric asymptotically.

Newman-Penrose: The conformal Weyl tensor $C^\mu{}_{\nu\lambda\rho}$ satisfies a decay condition: In suitably adapted spinor frames the essential components of the Weyl spinor show Sachs peeling behaviour:

$$\Psi_k = \frac{\Psi_k^0}{r^{5-k}} + o\left(\frac{1}{r^{5-k}}\right), \quad k = 0, 1, \ldots 4, \quad \text{as} \;\; r \to \infty,$$

with functions Ψ_k^0 which are r-independent and parametrized, as is $\mathcal{J}^+_*$, by the set of null generators of the congruence.

3 Underlying conformal geometry

Penrose developed the more geometrical concept of asymptotically simple space-times.

It assumes that the large scale conformal structure of the solution $(\hat{M}, \hat{g})$ admits a ('sufficiently') smooth conformal extension

$$\hat{M} \to M = \hat{M} \cup \mathcal{J}, \qquad \hat{g}_{\mu\nu} \to g_{\mu\nu} = \Omega^2 \, \hat{g}_{\mu\nu}$$

where M is smooth with boundary $\mathcal{J}$, $g_{\mu\nu}$, Ω are smooth on M,

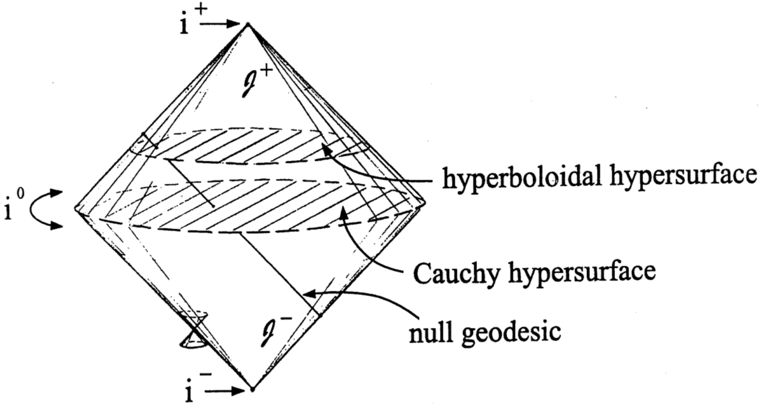
$$\Omega > 0 \text{ on } \hat{M}, \qquad \Omega = 0, \; d\Omega \neq 0 \text{ on } \mathcal{J}, \qquad \mathcal{J} = \mathcal{J}^- \cup \mathcal{J}^+,$$

with $\mathcal{J}^\mp \sim \mathbb{R} \times \mathbb{S}^2$ representing the past/future end points of null geodesics, i.e. past/future null infinity.

If $(\hat{M}, \hat{g})$ is vacuum near $\mathcal{J}^\mp$, the case considered in the following, the sets $\mathcal{J}^\mp$ are g-null hypersurfaces.
If (M, g) is sufficiently smooth its conformal Weyl tensor satisfies the Sachs peeling property.

4 The model . . .

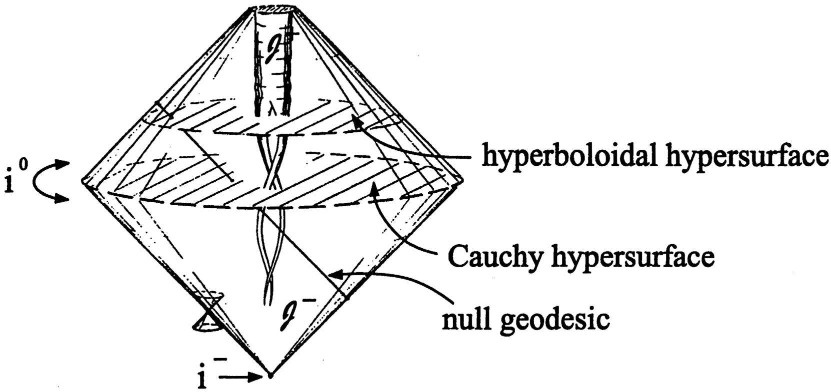
conformally compactified Minkowski space



$i^{\pm}$: $\Omega = 0, d\Omega = 0, Hess_g\Omega = -2\,g$ future/past time-like infinity,
i^0 : $\Omega = 0, d\Omega = 0, Hess_g\Omega = \quad 2\,g$ space-like infinity,

5 . . . and possible proposed generalizations

collapse generated purely by incoming gravitational radiation i.e. the radiation field Ψ_4^0 on $\mathcal{J}^-$ (in $\mathcal{J}^-$-adapted frame)



radiation from – inspiral – merger – ring down – of black holes, registered on $\mathcal{J}^+$ by the radiation field Ψ_4^0 (in $\mathcal{J}^+$-adapted frame).

6 Different sorts of existence results.

While the concept of asymptotic simplicity does not depend on specific coordinates, it is conveniently and most often analysed in terms of Bondi coordinates. The sets $\{u = u_* = const\} \cap \mathcal{J}^+ \sim \mathbb{S}^2$ are then space-like and $\mathcal{J}^+$ represents the future endpoints of the null geodesics ruling the null hypersurfaces $\{u = u_* = const\}$.

In the first 20 years the analysis of the new approach focussed on geometrical properties and physical concepts and was essentially based on formal asymptotic expansion type analyses in terms of r^{-k} with $k \in \mathbb{Z}$. Polyhomogeneous expansions in terms of $r^{-k} log^j r$ with $k, j \in \mathbb{Z}$, were also proposed but seriously pursued only later.

The existence results discussed in the following deal with solutions

– obtained by using the equations in terms of the conformal fields g and Ω, giving solutions with smooth conformal extensions,
– analysed in terms of the physical metric $\hat{g}$, allowing for more general data, giving weaker smoothness at null infinity.

7 Characteristic initial value problems I

Consider the following data:

- The radiation field Ψ_4^0 on future null infinity $\mathcal{J}^+$,
- the null datum Ψ_0 on a given outgoing null hypersurface $\mathcal{N}$, satisfying suitable fall-off conditions at $\Sigma = \mathcal{N} \cap \mathcal{J}^+ \sim \mathbb{S}^2$,
- certain pieces on Σ.

If these data are real analytic there exists a (unique) real analytic, 'asymptotically simple' solution in some neighbourhood of $\mathcal{J}^+$.

If these data are C^∞ there exists a (unique) 'asymptotically simple' C^∞ solution on some neighbourhood of Σ in the past of $\mathcal{N} \cup \mathcal{J}^+$.

To say more, further assumptions are needed. If extended into the interior, the hypersurface $\mathcal{N}$, smooth near Σ, develops in general complicated caustics and selfintersections unless the datum Ψ_0 is given such $\mathcal{N}$ can be identified with the part $\mathcal{C}_p^+ \setminus \{p\}$ of a cone $\mathcal{C}_p^+$ generated by the future directed null geodesics emanating from a regular point p in the solution.

8 Characteristic initial value problems II

Assume Ψ_0 is C^∞ on $\mathcal{C}_p^+ \setminus \{p\}$ and satisfies (the somewhat subtle) regularity conditions at p.

Then there exists a unique maximal globally hyperbolic smooth solution in the future of $\mathcal{C}_p^+$ which induces the given data.

If the data satisfy in addition suitable fall-off conditions near Σ (excluding, in particular, caustics in the future of p), the solution admits a piece $\mathcal{J}'^+$ of a smooth conformal boundary near Σ.

In this case the radiation field Ψ_4^0 on $\mathcal{J}'^+$ is determined by the solution, thus by the data on $\mathcal{C}_p^+$, and cannot be prescribed as independent data.

An extreme situation arises if we consider asymptotically simple solutions that admit conformal extensions with regular points i^- and we let $p \to i^-$ so that $\mathcal{J}^- \cup i^-$ can be identified with $\mathcal{C}_{i^-}^+$. The data to be considered on $\mathcal{J}^- \cup \{i^-\}$ is then the radiation field, given in a suitably $\mathcal{J}^-$-adapted frame by Ψ_4^0.

9 Asymptotic characteristic initial value problem

Assume that Ψ^0_4 is smooth on $\mathcal{J}^-$ in the usual sense and satisfies near i^- appropriate regularity conditions.

Then there exists a unique, smooth, maximal globally hyperbolic, past asymptotically simple solution in the future of $\mathcal{J}^- \cup \{i^-\}$ which induces the given (incoming) radiation field on this set.

If the radiation field vanishes near i^- the solution will be flat near i^-, regularity at i^- poses not problem, and the situation is covered in fact by the second of the results above.

The maximal globally hyperbolic developments of these solutions are under control so far only under additional assumptions.

There does exist a class of radiation fields so that $\mathcal{J}^-$ is complete and the corresponding solutions admit smooth conformal extensions with smooth, complete $\mathcal{J}^+$ and regular point i^+.

The way these solutions are constructed will be discussed later.

A general characterization of this class of data is not known yet.

10 Hyperboloidal initial value problem I

To avoid caustics, we consider hyperboloidal hypersurfaces, i.e. space-like hypersurfaces S which are thought to extend as space-like hypersurfaces up to $\mathcal{J}^+$ with space-like boundary $\Sigma = \partial S = S \cap \mathcal{J}^+ \sim \mathbb{S}^2$.

Conformal data induced by an asymptotically simple vacuum solution on a hyperboloidal hypersurface that extends smoothly to Σ are referred to as smooth hyperboloidal initial data.

The corresponding physical vacuum data $\hat{h}_{ab}$ and $\hat{\chi}_{ab}$ show a specific fall-off behaviour near Σ. If $\omega \geq 0$ is in $C^\infty(S)$ with $\omega = 0, d\omega \neq 0$ on Σ then, in particular,

− mean extrinsic curvature: $\lim_{\omega \to 0} \hat{h}^{ab}\,\hat{\chi}_{ab} \neq 0 \quad \text{on} \quad \Sigma$,

− $h_{ab} = \Omega^2 \hat{h}_{ab}$ and $\chi_{ab} = \Omega\,(\hat{\chi}_{ab} - \frac{1}{3}\,\hat{h}^{cd}\,\hat{\chi}_{cd}\,\hat{\chi}_{ab})$ extend smoothly to Σ with h_{ab} Riemannian and $\Omega \sim \omega$ as $\omega \to 0$.

11 Hyperboloidal initial value problem II

In general, free data $\hat{h}_{ab}$ and $\hat{\chi}_{ab}$ satisfying the conditions above yield constrained conformal data that are polyhomogeneous i.e. admit non-trivial asymptotic expansions in terms of $\omega^k \log^j \omega$ at Σ.

If the free data are only polyhomogeneous at Σ they determine constrained conformal data that are again polyhomogeneous at Σ.

There exists a large class of smooth hyperboloidal initial data. Obtained from free data satisfying additional conditions at Σ.

− Conservation of asymptotic smoothness: Smooth hyperboloidal vacuum data develop into vacuum solutions that admit in their future a smooth piece of $\mathcal{J}'^{+}$ ruled by g-null geodesics with past endpoints on Σ. Consequence of Einstein's field equations.

− Strong hyperboloidal stability: Data close to Minkowskian hyperboloidal data develop into solutions that admit smooth conformal extensions with regular i^+ so that $\mathcal{J}'^{+} \cup \{i^+\} = \mathcal{C}^{-}_{i^+}$.

− Choice ?: Evolution of polyhomogeneous data near null infinity.

12 Standard Cauchy problem: Choice of data I

Consider asymptotically flat standard Cauchy data $\hat{h}_{ab}$ and $\hat{\chi}_{ab}$ on $\mathbb{R}^3$. Instead of weighted Sobolev norms list only fall-off properties.

L. Bieri (2009): weakest fall-off

$$\hat{h}_{\alpha\beta} = \delta_{\alpha\beta} + o_3(|x|^{-1/2}) \qquad \hat{\chi}_{\alpha\beta} = o_2(|x|^{-3/2}).$$

D. Christodoulou, S. Klainerman (1993):

$$\hat{h}_{\alpha\beta} = \left(1 + 2\,m\,|x|^{-1}\right)\,\delta_{\alpha\beta} + o_4(|x|^{-3/2}) \qquad \hat{\chi}_{\alpha\beta} = o_3(|x|^{-5/2}).$$

Hintz-Vasy (2017): $0 < \epsilon << 1$

$$\hat{h}_{\alpha\beta} - (1 - \xi)\,\hat{h}^S_{\alpha\beta} = o(|x|^{-(1+\epsilon)}), \quad \hat{\chi}_{\alpha\beta} = o(|x|^{-(2+\epsilon)}),$$

$\hat{h}^S_{\alpha\beta}$ 3-metric induced by standard Schwarzschild metric on $\{t = 0, r \geq R\}$, and $\xi \in C_0^\infty(\mathbb{R}^3)$, $\xi = 1$ in B_R, $R > 0$.

These two authors consider in particular data that admit polyhomogeneous asymptotic expansion in terms of integer, fractional or even complex powers of $|x|^{-1}$ and logarithmic terms as $|x| \to \infty$.

13 Standard Cauchy problem: Choice of data II

Dain-F. (2001): Study data which are clean at space-like infinity, i.e. admit expansions in terms of integer powers of $|x|^{-1}$

$$\hat{h}_{\alpha\beta} = \left(1 + 2\, m\, |x|^{-1}\right)\, \delta_{\alpha\beta} + O_{\infty}(|x|^{-2}) \qquad \hat{\chi}_{\alpha\beta} = O_{\infty}(|x|^{-2}),$$

start from free data that admit $|x|^{-1}$-expansion, conformal 3-structure that admits smooth conformal compactification.

Free $\hat{\chi}_{\alpha\beta}$-data that fall off like $O(|x|^{-(2+\epsilon)})$ give rise to clean solutions to the constraints. The free $\hat{\chi}_{\alpha\beta}$-data which determine the linear ADM-momentum, namely those that fall off like $O(|x|^{-2})$ but not faster, give rise to logarithmic terms.

Compare with Bieri-, Christodoulou-Klainerman-, Hintz-Vasy-data.

Only example for which the coefficients entering polyhomogeneity expansion could admit some physical interpretation ?

What else could be required besides $O(|x|^{-k})$ fall-off conditions ? The type of data above seem to exhaust the possibilities.

14 Cauchy problem: Choice of data III

Corvino (2000), Chrusciel-Delay (2003), Corvino-Schoen (2006):

Construct deformations of given asymptotically flat data into data that are
– exactly static/stationary near space-like infinity, or
– asymptotically static/stationary at space-like infinity,
while keeping the given data unchanged on a prescribed compact set in the interior.

These are again data which are clean at space-like infinity. Near space-like infinity they are more special than the data considered above but they are still very general. How general they are will be seen later.

15 Cauchy problem: Results on evolution I

For data that are small, smooth, with fall-off conditions considered by the authors above, the global non-linear stability of Minkowski space and the following asymptotic behaviour has been established:

Bieri: As $r \to \infty$ along outgoing null geodesics in adapted frame

$$\Psi_k = O(r^{k-5}),\ k = 4, 3, \qquad \Psi_k = o(r^{-5/2}),\ k = 2, 1, 0.$$

Christodoulou-Klainerman:

$$\Psi_k = O(r^{k-5}),\ k = 4, 3, 2, \quad \Psi_k = O(r^{-7/2}),\ k = 1, 0.$$

No Sachs peeling or smooth conformal extensions of solutions for which these estimates are sharp. Restricted information on null infinity

Hintz-Vasy: For data that admit polyhomogeneous asymptotic expansion at space-like the solutions admit polyhomogeneous asymptotic expansion at space-like, null and time-like infinity.

16 Cauchy problem: Results on evolution II

Chruściel-Delay (2001), Corvino (2007):

Construct vacuum data with arbitrarily small mass $m > 0$ on $\mathbb{R}^3$ which are exactly Schwarzschild near space-like infinity.

As $m \to 0$ the developments in time of these data contain smooth hyperboloidal hypersurfaces with smooth induced data arbitrarily close to Minkowskian hyperboloidal data.

Combined with the strong hyperboloidal stability result above this implies the existence of a large class of vacuum solutions with complete, smooth null infinity $\mathcal{J}^{\pm}$ and regular points $i^{\pm}$ at past and future time-like infinity.

Can be expected to generalize to data which satisfy a similar smallness condition but which are only asymptotically static/ stationary at space-like infinity.

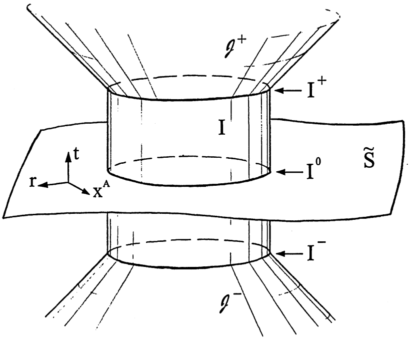
17 Space-like infinity in detail

Developments of data which are clean at space-like infinity admit near space-like infinity smooth, finite, conformal representations in which space-like infinity is given by a cylinder $I \sim]-1,1[\times\mathbb{S}^2$ that touches the Cauchy hypersurface in a set $I^0 \sim \mathbb{S}^2$ and null infinity $\mathcal{J}^\pm$ in the critical sets $I^\pm \sim \mathbb{S}^2$. No smallness conditions needed.

The setting and the gauge, including the conformal factor the coordinates and a g-orthonormal frame, are determined solely by the field equations, the conformal structure of the solutions, and by some gauge conditions on the initial slice. The location of the prospective hypersurfaces $\mathcal{J}^\pm$ is known explicitly in the given gauge.

The reduced conformal field equations are in this setting hyperbolic on $\hat{M} \cup I$ and, if the frame admits a continuous extension, also at null infinity $\mathcal{J}^\pm$.

18 Cylinder at space-like infinity



Suitably conformally rescaled data extend smoothly to $I^0 \sim \mathbb{S}^2$.

$I =$ cylinder at space-like infinity generated from I^0 by conformally extended Einstein equations, I touches $\mathcal{J}^\pm$ at critical sets $I^\pm \sim \mathbb{S}^2$

The reduced conformal Einstein equations are hyperbolic on $\tilde{M} \cup I \cup \mathcal{J}^\pm$, the hyperbolicity is lost at the critical sets $I^\pm$.

19 The critical sets

For their unknown u the reduced equations induce interior equations on I. These determine in terms of a coordinate $\rho \geq 0$, with $\rho = 0$ on I, a formal expansion

$$u \sim \sum_{k=0}^{\infty} \frac{1}{k!} c_k \, \rho^k \text{ with coefficients } c_k = (\partial_\rho^k u)_I \text{ on } I.$$

Even if the initial data are smooth near I^0, the solutions on I will in general not extend smoothly to $I^{\pm}$ (and thus to $\mathcal{J}^{\pm}$). While the $(\partial_\rho^k u)_I$ are smooth on I, they are polyhomogeneous in terms of a coordinate τ with $\tau = \pm 1$ at $I^{\pm}$: $(\partial_\rho^k u)_I \sim (1 \mp \tau)^j \log^l (1 \mp \tau)$.

Not a problem of the setting but a consequence of the evolution equations and the structure of the data.

The logarithmic terms depend on the choice of the Cauchy data. They vanish e.g. if the data are asymptotically static/stationary.

If the setting is linearized at Minkowski space the logarithmic terms travel along $\mathcal{J}^{\pm}$. The situation cannot be expected to improve but does not get worse in the non-linear case.

20 Polyhomogeneity of physical relevance ?

We can distinguish three sources of polyhomogeneous behaviour:

- Badly chosen coordinates may entail polyhomogeneities at $\mathscr{J}^{\pm}$ even for vacuum solutions with smooth conformal extensions.
- Certain data give rise to polyhomogeneities already on the initial slice at space-like infinity and then most likely along null infinity.
- For another class of data polyhomogeneities are generated at the critical sets and propagate along null infinity.

Polyhomogeneities $\leftrightarrow$ physical/geometrical concepts ?

Paetz (2018): If the radiation field on $\mathscr{J}^{\pm}$ vanishes at all orders at $I^{\pm}$ there are no logarithmic terms at the critical sets. With an additional condition on the mass aspects this condition is also necessary for the smoothness at the critical sets.

Going towards some interpretation of some polyhomogeneities and suggests conditions for solutions determined from data on $\mathscr{J}^{-} \cup i^{-}$ to extend near the cylinder at space-like infinity smoothly to $\mathscr{J}^{+}$.

21 Density results for vacuum data and solutions ?

Allen-Stavrov Allen (2017): Polyhomogeneous, hyperboloidal data of constant mean curvature can be approximated in certain Hölder norms by smooth hyperboloidal data of constant mean curvature.

Corvino-Schoen (2006): Asymptotically flat initial data on a 3-manifold S can be approximated by data on S which agree with the original data inside a given compact domain, and agree with some space-like Schwarzschild, Kerr or ... slice outside some ball.

Approximations controlled in terms of weighted Sobolev norms consistent with Christodoulou-Klainerman data. Generalizations ?

Raises the most interesting question:

Are the asymptotically simple vacuum solutions in some sense dense in a set of asymptotically flat vacuum solutions considered in the non-linear stability results above ?

A definite answer should provide essential clarification of situation.

22 References

A list of the authors of the work considered above is found in

H. Friedrich. Peeling or not peeling – is that the question ?
Class. Quantum Grav. 35 (2018) 083001.
arXiv: 1709.07709

which discusses the subject in more detail. The following articles appeared after the article above was posted on the arXiv

P. Hintz, A. Vasy. A global analysis proof of the stability of Minkowski space and the polyhomogeneity of the metric.
arXiv: 1711.00195

T. Paetz. On the smoothness of the critical sets of the cylinder at spatial infinity in vacuum spacetimes.
arXiv: 1804.05034