Scattering Equations and Soft Theorems

Ellis Ye Yuan

Institute for Advanced Study

Infrared Physics
International Solvay Institutes
May 16, 2018

Describe a formalism for
the scattering of *massless* particles,
and discuss its connection to
various soft theorems.

Parke–Taylor formula

Scattering of n gluons

$$\mathcal{M}_n = \mathrm{tr}(T_1 T_2 \cdots T_n) M_n[12 \ldots n] + \text{permutations.}$$

In the maximally-helicity-violating sector (MHV, two "$-$" helicities)

$$M_n^{\mathrm{MHV}}[12 \ldots n] = \frac{\langle i\, j \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle},$$

with $k_{a,\mu} \sigma^{\mu}_{\alpha\dot{\alpha}} \equiv \lambda_{a,\alpha} \tilde{\lambda}_{a,\dot{\alpha}}$, and $\langle a\, b \rangle \equiv \epsilon^{\alpha\beta} \lambda_{a,\alpha} \lambda_{b,\beta}$. [Parke, Taylor, '86]

Soft limit $k_n \to 0$ (assuming n is neither i nor j)

$$M_n^{\mathrm{MHV}}[12 \ldots n] = \frac{\langle n-1\, 1 \rangle}{\langle n-1\, n \rangle \langle n1 \rangle} \times M_{n-1}^{\mathrm{MHV}}[12 \ldots n-1] + \text{subleading.}$$

Parke–Taylor formula

$$M_n^{\mathrm{MHV}}[12\ldots n] = \frac{\langle i\, j\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle},$$

with $k_{a,\mu}\sigma^{\mu}_{\alpha\dot\alpha} \equiv \lambda_{a,\alpha}\tilde\lambda_{a,\dot\alpha}$, and $\langle a\, b\rangle \equiv \epsilon^{\alpha\beta}\lambda_{a,\alpha}\lambda_{b,\beta}$.

We can consistently assign for every particle

$$\lambda = \begin{pmatrix} 1 \\ z \end{pmatrix}, \qquad z \equiv z(k) = \frac{k^1 + ik^2}{k^0 + k^3},$$

and then

$$M_n^{\mathrm{MHV}}[12\ldots n] \propto \frac{1}{(z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)}.$$

z parametrizes the celestial sphere in four dimensions.

A slight detour ...

Here we would rather interpret this sphere as and auxiliary space, i.e.,

$$\boxed{M = \sum_{z(k)} I(z)\big|_{z=z(k)} = \underbrace{\int \mathrm{d}z\, \delta(z - z(k))\, I(z)}_{\text{“path integral”}}.}$$

For MHV gluon partial amplitudes

$$I(z) \propto \frac{1}{(z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)}, \qquad z_a(k) = \frac{k_a^1 + ik_a^2}{k_a^0 + k_a^3}.$$

Past 3/2 decades: twistor strings, ambi-twistor strings, etc.

[Nair, ‘88], [Witten, ‘03], [Mason, Skinner, ‘13], etc. [See Lionel’s talk.]

This exists for arbitrary tree-level scattering of massless particles.

[Cachazo, He, Yuan, ‘13]

In this talk we will directly work with this formalism.

The CHY formalism

For *n*-particle scattering, consider an *n*-punctured Riemann sphere, with $\{z_1, z_2, \ldots, z_n\}$ being the inhomogeneous coordinates of the punctures.

$$\{z_1, \ldots, z_n\} \sim \{\phi(z_1), \ldots, \phi(z_n)\}, \qquad \phi \in \mathrm{SL}(2, \mathbb{C}) : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}.$$

The CHY formalism

For n-particle scattering, consider an n-punctured Riemann sphere, with $\{z_1, z_2, \ldots, z_n\}$ being the inhomogeneous coordinates of the punctures.

$$\{z_1, \ldots, z_n\} \sim \{\phi(z_1), \ldots, \phi(z_n)\}, \qquad \phi \in \mathrm{SL}(2, \mathbb{C}) : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}.$$

CHY formalism

$$\boxed{A_n(\{k, \epsilon\}) = \int \underbrace{\frac{\mathrm{d}^n z}{\omega} {\prod_a}' \delta(f_a)}_{\mathrm{d}\mu_n} I_n(\{z\}, \{k, \epsilon\}).}$$

Scattering equations

$$\boxed{f_a \equiv \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{k_a \cdot k_b}{z_a - z_b} = 0, \qquad \forall a.}$$

The CHY formalism

$$A_n(\{k,\epsilon\}) = \int \underbrace{\frac{\mathrm{d}^n z}{\omega} {\prod}'_a \delta(f_a)}_{\mathrm{d}\mu_n} I_n(\{z\},\{k,\epsilon\}), \quad f_a \equiv \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{z_a - z_b}.$$

- For any choices of labels $\{i,j,k\}$ and $\{i',j',k'\}$

$$\omega \equiv (z_i - z_j)(z_j - z_k)(z_k - z_i)\mathrm{d}z_i \mathrm{d}z_j \mathrm{d}z_k,$$

$${\prod}'_a \equiv \frac{1}{(z_{i'} - z_{j'})(z_{j'} - z_{k'})(z_{k'} - z_{i'})} \prod_{a\neq i',j',k'} .$$

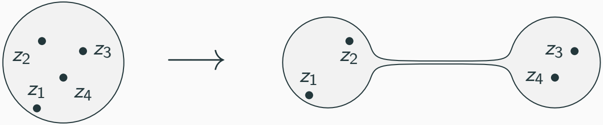
- Only I_n depends on the theory under study.
- $\mathrm{SL}(2,\mathbb{C})$ redundancy imposes a constraint on I_n

$$I_n \longmapsto I_n \prod_{a=1}^{n} (\gamma z_a + \delta)^4.$$

How to ensure unitarity

$$\{k_a\} \xrightarrow{\{f_a=0\}} \{z_a = z_a(\{k\})\}.$$

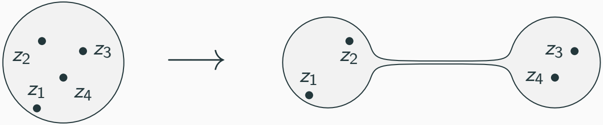
Factorization limit:

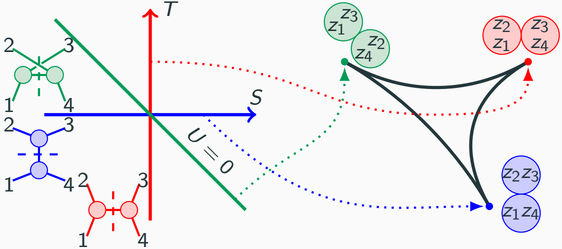


How to ensure unitarity

$$\{k_a\} \xrightarrow{\{f_a=0\}} \{z_a = z_a(\{k\})\}.$$

Factorization limit:





Compact formulas

Gluon amplitudes

$$\boxed{I_n^{\mathsf{YM}}[12\ldots c] = C[12\ldots n]\,\mathrm{Pf}'\Psi_n,}$$

where

$$C[12\ldots n] = \left((z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)\right)^{-1},$$

Compact formulas

Gluon amplitudes

$$\boxed{I_n^{\mathrm{YM}}[12\ldots c] = C[12\ldots n]\,\mathrm{Pf}'\Psi_n,}$$

where

$$C[12\ldots n] = \left((z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)\right)^{-1},$$

Ψ_n is $2n \times 2n$ anti-symmetric

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} 1\,2\,\cdots\,n \quad 1\,2\,\cdots\,n \\ \left(\begin{array}{c|c} A_n & -C_n^{\mathrm{T}} \\ \hline C_n & B_n \end{array}\right) \end{array}$$

Compact formulas

Gluon amplitudes

$$\boxed{I_n^{\mathsf{YM}}[12\ldots c] = C[12\ldots n]\,\mathrm{Pf}'\Psi_n,}$$

where

$$C[12\ldots n] = \left((z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)\right)^{-1},$$

Ψ_n is $2n \times 2n$ anti-symmetric

$$\begin{array}{cc} & \begin{array}{cc} 1\,2\,\cdots\,n & 1\,2\,\cdots\,n \end{array} \\ \begin{array}{c} 1\\ 2\\ \vdots\\ n\\ 1\\ 2\\ \vdots\\ n \end{array} & \left(\begin{array}{c|c} \frac{k_a\cdot k_b}{z_a-z_b} & \frac{k_a\cdot\epsilon_b}{z_a-z_b} \\ \hline \frac{\epsilon_a\cdot k_b}{z_a-z_b} & \frac{\epsilon_a\cdot\epsilon_b}{z_a-z_b} \end{array}\right) \end{array}$$

Compact formulas

Gluon amplitudes

$$\boxed{I_n^{\mathrm{YM}}[12\ldots c] = C[12\ldots n]\,\mathrm{Pf}'\Psi_n,}$$

where

$$C[12\ldots n] = \left((z_1-z_2)(z_2-z_3)\cdots(z_n-z_1)\right)^{-1},$$

Ψ_n is $2n \times 2n$ anti-symmetric

$$\begin{array}{cc}
 & \begin{array}{cc} 1\,2\,\cdots\,n & \quad 1\,2\,\cdots\,n \end{array} \\
\begin{array}{c}1\\2\\\vdots\\n\\1\\2\\\vdots\\n\end{array} & \left(\begin{array}{c|c} \frac{k_a\cdot k_b}{z_a-z_b} & \frac{k_a\cdot\epsilon_b}{z_a-z_b} \\ \hline \frac{\epsilon_a\cdot k_b}{z_a-z_b} & \frac{\epsilon_a\cdot\epsilon_b}{z_a-z_b} \end{array}\right)
\end{array}$$

Compact formulas

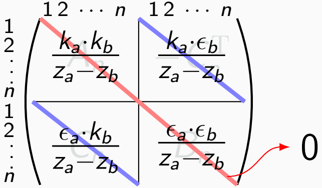
Gluon amplitudes

$$\boxed{I_n^{\mathrm{YM}}[12\ldots c] = C[12\ldots n]\,\mathrm{Pf}'\Psi_n,}$$

where

$$C[12\ldots n] = \left((z_1-z_2)(z_2-z_3)\cdots(z_n-z_1)\right)^{-1},$$

Ψ_n is $2n \times 2n$ anti-symmetric



$$\begin{array}{cc} & \begin{array}{cc} 1\,2\,\cdots\,n & 1\,2\,\cdots\,n \end{array} \\ \begin{array}{c}1\\2\\\vdots\\n\\1\\2\\\vdots\\n\end{array} & \left(\begin{array}{c|c} \frac{k_a\cdot k_b}{z_a-z_b} & \frac{k_a\cdot\epsilon_b}{z_a-z_b} \\ \hline \frac{\epsilon_a\cdot k_b}{z_a-z_b} & \frac{\epsilon_a\cdot\epsilon_b}{z_a-z_b}\end{array}\right) \end{array} \rightarrow 0$$

Compact formulas

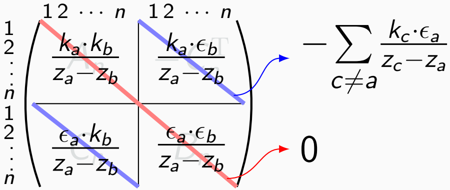
Gluon amplitudes

$$\boxed{I_n^{\mathrm{YM}}[12\ldots c] = C[12\ldots n]\,\mathrm{Pf}'\Psi_n,}$$

where

$$C[12\ldots n] = \left((z_1-z_2)(z_2-z_3)\cdots(z_n-z_1)\right)^{-1},$$

Ψ_n is $2n \times 2n$ anti-symmetric



$$\begin{array}{c} \\ \begin{matrix}1\\2\\ \vdots\\ n\\ 1\\2\\ \vdots\\ n\end{matrix}\end{array} \begin{array}{c} \begin{matrix}1\,2\,\cdots\,n & \quad 1\,2\,\cdots\,n\end{matrix} \\ \left(\begin{array}{c|c} \frac{k_a\cdot k_b}{z_a-z_b} & \frac{k_a\cdot\epsilon_b}{z_a-z_b} \\ \hline \frac{\epsilon_a\cdot k_b}{z_a-z_b} & \frac{\epsilon_a\cdot\epsilon_b}{z_a-z_b} \end{array}\right) \end{array}$$

$$-\sum_{c\neq a}\frac{k_c\cdot\epsilon_a}{z_c-z_a}$$

$$0$$

Compact formulas

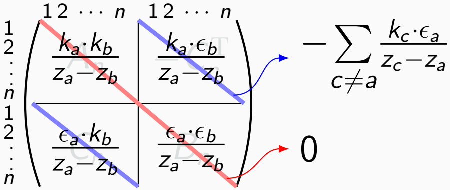
Gluon amplitudes

$$\boxed{I_n^{\mathrm{YM}}[12\ldots c] = C[12\ldots n]\,\mathrm{Pf}'\Psi_n,}$$

where

$$C[12\ldots n] = ((z_1-z_2)(z_2-z_3)\cdots(z_n-z_1))^{-1}\,,$$

Ψ_n is $2n\times 2n$ anti-symmetric



$$\begin{array}{c} \\ \begin{matrix}1\\2\\\vdots\\n\\1\\2\\\vdots\\n\end{matrix}\end{array}\begin{array}{c}\begin{matrix}1\,2\,\cdots\,n & 1\,2\,\cdots\,n\end{matrix}\\ \left(\begin{array}{c|c} \frac{k_a\cdot k_b}{z_a-z_b} & \frac{k_a\cdot\epsilon_b}{z_a-z_b} \\ \hline \frac{\epsilon_a\cdot k_b}{z_a-z_b} & \frac{\epsilon_a\cdot\epsilon_b}{z_a-z_b}\end{array}\right)\end{array}$$

$$-\sum_{c\neq a}\frac{k_c\cdot\epsilon_a}{z_c-z_a}$$

$$0$$

$$\mathrm{Pf}'\Psi_n = \frac{(-1)^{a,b}}{z_a-z_b}\mathrm{Pf}(\Psi_n)_{\hat{a},\hat{b}:}.$$

Compact formulas

Graviton amplitudes

$$\boxed{I_n^{\mathsf{GR}} = (\mathrm{Pf}'\Psi_n)^2 \equiv \mathsf{det}'\Psi_n.}$$

Compact formulas

Graviton amplitudes

$$\boxed{I_n^{\mathsf{GR}} = (\mathrm{Pf}'\Psi_n)^2 \equiv \mathsf{det}'\Psi_n.}$$

We can also utilize the first block of Ψ_n, i.e., $(A_n)_{ab} = \frac{k_a \cdot k_b}{z_a - z_b}$.

Compact formulas

Graviton amplitudes

$$\boxed{I_n^{\mathsf{GR}} = (\mathrm{Pf}'\Psi_n)^2 \equiv \det{}'\Psi_n.}$$

We can also utilize the first block of Ψ_n, i.e., $(A_n)_{ab} = \frac{k_a \cdot k_b}{z_a - z_b}$.

This leads to formulas for a bunch of other effective theories, e.g.,

- $U(N)$ Non-linear sigma model

$$I_n^{\mathsf{NLSM}}[12\ldots n] = C[12\ldots n]\,(\mathrm{Pf}'A_n)^2.$$

- Born–Infeld

$$I_n^{\mathsf{BI}} = (\mathrm{Pf}'A_n)^2\,\mathrm{Pf}'\Psi_n.$$

- A special Galileon scalar

$$I_n^{\mathsf{sGal}} = (\mathrm{Pf}'A_n)^4.$$

More: [Cachazo, He, Yuan, '14], [He, Zhang, '16], [Heydeman, Schwarz, Wen, '17], etc.

More about the scattering equations

For any kinematics $\{k\}$, these always yield $(n-3)!$ solutions in $\{z\}$.

Soft limit, $k_n \to 0$:

- For $1 \leq a < n$, the equations reduce to those of $n-1$ particles

$$\sum_{b \neq a}^{n-1} \frac{k_a \cdot k_b}{z_a - z_b} \cancel{+ \frac{k_a \cdot k_n}{z_a - z_n}} = 0.$$

- For each solution of $\{z_1, \ldots, z_{n-1}\}$ from above, the last equation yields $n-3$ solutions for z_n (due to $\sum_{a=1}^{n} k_a = 0$).

$$\sum_{b=1}^{n-1} \frac{k_n \cdot k_b}{z_n - z_b} = 0.$$

- In all solutions z_n is distinct from the other z_a's.

Single soft particle

$$\int \delta(f) \cdots \longrightarrow \oint\limits_{|f|=\epsilon} \frac{1}{f} \cdots .$$

Soft limit, $k_n \to 0$:

$$M_n = \int \underbrace{\frac{\mathrm{d}^n z}{\omega} {\prod_a}' \delta(f_a)}_{\mathrm{d}\mu_n} I_n = \int \mathrm{d}\mu_{n-1} \oint\limits_{|f_n|=\epsilon} \frac{\mathrm{d}z_n}{f_n} I_n + \text{subleading.}$$

Applying residue theorem on z_n

$$\boxed{M_n = \int \mathrm{d}\mu_{n-1} \underbrace{\oint\limits_{\substack{\text{double poles} \\ \text{in } I_n}} \frac{\mathrm{d}z_n}{\sum_{b=1}^{n-1} \frac{k_n \cdot k_b}{z_n - z_b}} I_n}_{\mathcal{S}^{(0)} I_{n-1}} + \text{subleading.}}$$

Convenient if a closed formula for all amplitudes in a theory is known.

Single soft particle

$$A_n = \int \mathrm{d}\mu_{n-1} \underbrace{\oint_{\substack{\text{double poles}\\ \text{in } I_n}} \frac{\mathrm{d}z_n}{\sum_{b=1}^{n-1} \frac{k_n \cdot k_b}{z_n - z_b}} I_n}_{\mathcal{S}^{(0)} I_{n-1}} + \text{(subleading)}.$$

- In CHY, soft theorems are essentially consequences of residue theorems.
- Before contour deformation, we have a drastically different expansion of the soft limit, in terms of the $n-3$ solutions of z_n.
- We expect each term in the soft factor to arise from collision of punctures $z_n \to z_b$.

Weinberg soft theorems

- Parke–Taylor factor

$$C[12\ldots n] = \frac{(z_{n-1} - z_1)}{(z_{n-1} - z_n)(z_n - z_1)}\, C[12\ldots n{-}1].$$

Weinberg soft theorems

- Parke–Taylor factor

$$C[12\dots n] = \frac{(z_{n-1} - z_1)}{(z_{n-1} - z_n)(z_n - z_1)}\, C[12\dots n{-}1].$$

- Recall the recursion formula for Pfaffian

$$\mathrm{Pf}\, M = \sum_{j\neq i} (-1)^{i+j+1+\theta(i-j)} M_{i\,j}\, \mathrm{Pf}\, M_{\hat{i}\,\hat{j}}.$$

In the n^{th} row of Ψ_n, all entries scale with k_n except for $(\Psi_n)_{n,2n}$.

$$\left(\frac{k_n \cdot k_1}{z_n - z_1}, \dots, \frac{k_n \cdot k_{n-1}}{z_n - z_{n-1}}, 0 \;\middle|\; \frac{k_n \cdot \epsilon_1}{z_n - z_1}, \dots, \frac{k_n \cdot \epsilon_{n-1}}{z_n - z_{n-1}}, \underbrace{\sum_{b=1}^{n-1} \frac{\epsilon_n \cdot k_b}{z_n - z_b}}_{\text{dominate}} \right).$$

Weinberg soft theorems

- Parke–Taylor factor

$$C[12\dots n] = \frac{(z_{n-1} - z_1)}{(z_{n-1} - z_n)(z_n - z_1)}\, C[12\dots n-1].$$

- Recall the recursion formula for Pfaffian

$$\mathrm{Pf}\, M = \sum_{j\neq i} (-1)^{i+j+1+\theta(i-j)} M_{i\,j}\, \mathrm{Pf}\, M_{\hat{i}\,\hat{j}}.$$

In the n^{th} row of Ψ_n, all entries scale with k_n except for $(\Psi_n)_{n,2n}$.

$$\mathrm{Pf}'\Psi_n = \underbrace{\left(\sum_{b=1}^{n-1} \frac{\epsilon_n \cdot k_b}{z_n - z_b}\right)}_{(\Psi_n)_{n,2n}} \mathrm{Pf}' \underbrace{(\Psi_n)_{\hat{n},\widehat{2n}}}_{\substack{\Psi_{n-1} \\ \text{no dependence} \\ \text{on } k_n}} + \text{subleading}.$$

Weinberg soft theorem

$$C[12\ldots n] = \frac{(z_{n-1}-z_1)}{(z_{n-1}-z_n)(z_n-z_1)}\, C[12\ldots n-1],$$

$$\mathrm{Pf}'\Psi_n = \left(\sum_{b=1}^{n-1} \frac{\epsilon_n\cdot k_b}{z_n - z_b}\right)\mathrm{Pf}'\Psi_{n-1} + \text{subleading.}$$

Gluons:

$$M_n^{\mathsf{YM}}[12\ldots n] \to \int \mathrm{d}\mu_{n-1} I_{n-1}^{\mathsf{YM}}[12\ldots n-1]\times$$

$$\oint \frac{\mathrm{d}z_n}{\sum_{b=1}^{n-1}\frac{k_n\cdot k_b}{z_n-z_b}}\, \frac{(z_{n-1}-z_1)}{(z_{n-1}-z_n)(z_n-z_1)} \left(\sum_{b=1}^{n-1}\frac{\epsilon_n\cdot k_b}{z_n-z_b}\right)$$

$$= \left(\frac{\epsilon_n\cdot k_1}{k_n\cdot k_1} - \frac{\epsilon_n\cdot k_{n-1}}{k_n\cdot k_{n-1}}\right) M_{n-1}^{\mathsf{YM}}[12\ldots n-1].$$

Weinberg soft theorem

$$C[12\ldots n] = \frac{(z_{n-1}-z_1)}{(z_{n-1}-z_n)(z_n-z_1)}\, C[12\ldots n-1],$$

$$\mathrm{Pf}'\Psi_n = \left(\sum_{b=1}^{n-1} \frac{\epsilon_n \cdot k_b}{z_n - z_b}\right) \mathrm{Pf}'\Psi_{n-1} + \text{subleading}.$$

Gravitons:

$$M_n^{\mathrm{GR}} \to \int \mathrm{d}\mu_{n-1}\, I_{n-1}^{\mathrm{GR}} \times$$

$$\oint \frac{\mathrm{d}z_n}{\sum_{b=1}^{n-1} \frac{k_n \cdot k_b}{z_n - z_b}} \left(\sum_{b=1}^{n-1} \frac{\epsilon_n \cdot k_b}{z_n - z_b}\right)^2$$

$$= \left(\sum_{b=1}^{n-1} \frac{\epsilon_{n,\mu\nu} k_b^\mu k_b^\nu}{k_n \cdot k_b}\right) M_{n-1}^{\mathrm{GR}}.$$

Extention of theories from the soft limit

- By simple power counting using either Feynman diagrams or CHY, soft limit of, e.g. NLSM, is suppressed

$$\mathrm{d}\mu_n \sim k_n^{-1},\ C \sim k_n^0,\ \mathrm{Pf}' A_n \sim k_1 \implies \int \mathrm{d}\mu_n C (\mathrm{Pf}' A_n)^2 \sim k_n.$$

Extention of theories from the soft limit

- By simple power counting using either Feynman diagrams or CHY, soft limit of, e.g. NLSM, is suppressed

$$\mathrm{d}\mu_n \sim k_n^{-1},\ C \sim k_n^0,\ \mathrm{Pf}' A_n \sim k_1 \implies \int \mathrm{d}\mu_n C (\mathrm{Pf}' A_n)^2 \sim k_n.$$

- Nevertheless, check the leading order

$$\mathrm{Pf}' A_n = \frac{(-1)^n}{z_1 - z_{n-1}} \sum_{b=2}^{n-2} (-1)^b \frac{k_n \cdot k_b}{z_n - z_b} \mathrm{Pf}(A_n)_{\hat{1},\hat{b},\widehat{n-1},\hat{n}}.$$

Extention of theories from the soft limit

- By simple power counting using either Feynman diagrams or CHY, soft limit of, e.g. NLSM, is suppressed

$$\mathrm{d}\mu_n \sim k_n^{-1},\ C \sim k_n^0,\ \mathrm{Pf}' A_n \sim k_1 \implies \int \mathrm{d}\mu_n C(\mathrm{Pf}' A_n)^2 \sim k_n.$$

- Nevertheless, check the leading order

$$\mathrm{Pf}' A_n = \frac{(-1)^n}{z_1 - z_{n-1}} \sum_{b=2}^{n-2} (-1)^b \frac{k_n \cdot k_b}{z_n - z_b} \mathrm{Pf}(A_n)_{\hat{1},\hat{b},\widehat{n-1},\hat{n}}.$$

From similar computations, $M_n^{\mathrm{NLSM}}[1 \ldots n]$ becomes

$$\sum_{b=2}^{n-2} k_n \cdot k_b \int \mathrm{d}\mu_{n-1}\, C[1 \ldots n-1]\, C[1\, a\, n-1] (\mathrm{Pf}(A_{n-1})_{\hat{1},\hat{a},\widehat{h-1}})^2.$$

Extention of theories from the soft limit

$$\int \mathrm{d}\mu_{n-1}\, C[1\ldots n{-}1]\ \underbrace{C[1\, a\, n{-}1]}_{\substack{\text{extra particle}\\ \text{extra flavor}}}(\mathrm{Pf}(A_{n-1})_{\hat{1},\hat{a},\widehat{h-1}})^2.$$

Extention of theories from the soft limit

$$\int \mathrm{d}\mu_{n-1}\, C[1\ldots n{-}1]\; \underbrace{C[1\, a\, n{-}1]}_{\substack{\text{extra particle}\\ \text{extra flavor}}}(\mathrm{Pf}(A_{n-1})_{\hat{1},\hat{a},\widehat{h-1}})^2.$$

This is a special case to the more general formula

$$\boxed{A_n^{\mathrm{NLSM}\oplus\Phi^3}[\alpha;\beta] = \int \mathrm{d}\mu_n\, C[\alpha]\, C[\beta]\, (\mathrm{Pf}(A_n)_{\hat{\beta}})^2.}$$

[Cachazo, Cha, Mizera, ‘16]

Extention of theories from the soft limit

$$\int \mathrm{d}\mu_{n-1}\, C[1\ldots n{-}1]\; \underbrace{C[1\, a\, n{-}1]}_{\substack{\text{extra particle}\\ \text{extra flavor}}}(\mathrm{Pf}(A_{n-1})_{\hat{1},\hat{a},\widehat{h-1}})^2.$$

This is a special case to the more general formula

$$\boxed{A_n^{\mathrm{NLSM}\oplus\Phi^3}[\alpha;\beta] = \int \mathrm{d}\mu_n\, C[\alpha]\, C[\beta]\, (\mathrm{Pf}(A_n)_{\hat{\beta}})^2.}$$

[Cachazo, Cha, Mizera, ‘16]

- Sector 1: the original $U(N)$ NLSM

$$A_n^{\mathrm{NLSM}\oplus\Phi^3}[\alpha;\varnothing] \equiv A_n^{\mathrm{NLSM}}[\alpha] = \int \mathrm{d}\mu_n\, C[\alpha]\, (\mathrm{Pf}' A_n)^2.$$

- Sector 2: bi-flavored Φ w/ $f_{abc} f_{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'}$

$$A_n^{\mathrm{NLSM}\oplus\Phi^3}[\alpha;\beta] \equiv A_n^{\Phi^3}[\alpha;\beta] = \int \mathrm{d}\mu_n\, C[\alpha]\, C[\beta], \qquad \beta \in S_n.$$

Soft theorems at subleading orders

- A revival of studying soft theorems at subleading orders arises from conjectured connections to the BMS symmetry. [Strominger, '13, '14] In particular

$$MA_n^{\mathrm{GR}} = (\mathcal{S}_{\mathrm{GR}}^{(0)} + \mathcal{S}_{\mathrm{GR}}^{(1)} + \mathcal{S}_{\mathrm{GR}}^{(2)}) M_{n-1}^{\mathrm{GR}} + \mathcal{O}(k_n^2).$$

- Using CHY formulas, these can be investigated by working into higher orders in k_n; the role of the residue theorem remains the same (f_n only receives an overall scale).

  E.g., [Schwab, Volovich, '14], [Afkhami-Jeddi, '14], [Zlotnikov, '14], etc.

- Note that at higher orders we necessarily need to expand $\delta(f_a)$ as well. In fact, the orbital part of $S^{(1)}$ can almost be easily read off by matching the $\delta'(f_a)$ terms.

Two soft particles

- When the single soft limit is suppressed, the limit of two simultaneous soft particles usually becomes interesting.

$$M^{\text{NLSM}} \sim k_n, \quad M^{\text{DBI}} \sim k_n^2, \quad M^{\text{sGal}} \sim k_n^3.$$

- Let us start with $n+2$ particles and control the double soft limit by

$$k_{n+1} = \tau\, p, \quad k_{n+2} = \tau\, q, \quad \tau \to 0.$$

- Clearly we have to integrate away z_{n+1} and z_{n+2} in order to land on an n-particle scattering. It is convenient to redefine

$$z_{n+1} = \rho - \frac{\xi}{2}, \qquad z_{n+2} = \rho + \frac{\xi}{2}.$$

Two soft particles

- $\{f_a = 0\}$ $(a \leq n)$ at leading order reduce to the scattering equations at lower points (independent of $\{\rho, \xi\}$).
- Instead of directly using $f_{n+1} = f_{n+2} = 0$, we impose

$$\underbrace{f_{n+1} + f_{n+2}}_{\rho \text{ contour}} = \underbrace{f_{n+1} - f_{n+2}}_{\text{solve } \xi} = 0.$$

Two soft particles

- $\{f_a = 0\}$ ($a \leq n$) at leading order reduce to the scattering equations at lower points (independent of $\{\rho, \xi\}$).
- Instead of directly using $f_{n+1} = f_{n+2} = 0$, we impose

$$\underbrace{f_{n+1} + f_{n+2}}_{\rho \text{ contour}} = \underbrace{f_{n+1} - f_{n+2}}_{\text{solve } \xi} = 0.$$

Correspondingly

$$M_{n+2} = \oint \mathrm{d}\rho \sum_{\xi \text{ solns}} \int \mathrm{d}\mu_n \frac{1}{f_{n+1} + f_{n+2}} \frac{-2}{\partial_\xi (f_{n+1} - f_{n+2})} I_{n+2}.$$

Two soft particles

- $\{f_a = 0\}$ ($a \leq n$) at leading order reduce to the scattering equations at lower points (independent of $\{\rho, \xi\}$).
- Instead of directly using $f_{n+1} = f_{n+2} = 0$, we impose

$$\underbrace{f_{n+1} + f_{n+2}}_{\rho \text{ contour}} = \underbrace{f_{n+1} - f_{n+2}}_{\text{solve } \xi} = 0.$$

Correspondingly

$$M_{n+2} = \oint \mathrm{d}\rho \sum_{\xi \text{ solns}} \int \mathrm{d}\mu_n \frac{1}{f_{n+1} + f_{n+2}} \frac{-2}{\partial_\xi (f_{n+1} - f_{n+2})} I_{n+2}.$$

Here some new phenomenon occurs.

Two soft particles

- $f_{n+1} - f_{n+2} = 0$ is equivalent to

$$\sum_{b=1}^{n} \left(\frac{k_b \cdot p}{\rho - \xi/2 - z_b} - \frac{k_b \cdot q}{\rho + \xi/2 - z_b} \right) - \frac{2\tau\, p \cdot q}{\xi} = 0.$$

Two soft particles

- $f_{n+1} - f_{n+2} = 0$ is equivalent to

$$\sum_{b=1}^{n} \left(\frac{k_b \cdot p}{\rho - \xi/2 - z_b} - \frac{k_b \cdot q}{\rho + \xi/2 - z_b} \right) - \frac{2\tau \, p \cdot q}{\xi} = 0.$$

- There are two types of ξ solutions:
  - Regular: $\xi \sim \tau^0$.
  - Singular: $\xi = \tau \xi_1 + \mathcal{O}(\tau^2)$. ξ_1 is uniquely solved.

Two soft particles

- $f_{n+1} - f_{n+2} = 0$ is equivalent to

$$\sum_{b=1}^{n} \left(\frac{k_b \cdot p}{\rho - \xi/2 - z_b} - \frac{k_b \cdot q}{\rho + \xi/2 - z_b} \right) - \frac{2\tau\, p \cdot q}{\xi} = 0.$$

- There are two types of ξ solutions:
  - Regular: $\xi \sim \tau^0$.
  - Singular: $\xi = \tau \xi_1 + \mathcal{O}(\tau^2)$. ξ_1 is uniquely solved.
- It turns out that for many theories of interests, singular solutions dominates over regular ones.

  Consequently

$$\boxed{M_{n+2} = \int \mathrm{d}\mu_n \oint \mathrm{d}\rho \, \frac{1}{\sum_{b=1}^{n} \frac{k_b \cdot (p+q)}{\rho - z_b}} \frac{\xi_1^2}{\tau\, p \cdot q} \, I_{n+2} + \text{subleading.}}$$

Examples of double soft theorems

$$A_n = \left(\begin{array}{c:cc} A_{n-2} & \vdots & \vdots \\ \hdashline \cdots & 0 & \frac{\tau^2\, p\cdot q}{-\xi} \\ \cdots & \frac{\tau^2\, p\cdot q}{\xi} & 0 \end{array}\right) \quad \Rightarrow \quad \mathrm{Pf}' A_n = \frac{\tau\, p\cdot q}{\xi_1}\, \mathrm{Pf}' A_{n-2} + \mathcal{O}(\tau^2).$$

Examples of double soft theorems

$$A_n = \left(\begin{array}{c:cc} A_{n-2} & \vdots & \vdots \\ \hdashline \dots & 0 & \frac{\tau^2\, p\cdot q}{-\xi} \\ \dots & \frac{\tau^2\, p\cdot q}{\xi} & 0 \end{array}\right) \quad \Rightarrow \quad \mathrm{Pf}' A_n = \frac{\tau\, p\cdot q}{\xi_1}\, \mathrm{Pf}' A_{n-2} + \mathcal{O}(\tau^2).$$

For sGal, DBI, EMS (with $m = 1, 0, -1$)

$$M_{n+2} = (k_{n+1} \cdot k_{n+2})^m (\mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \mathcal{S}^{(2)}) M_n + \mathcal{O}(\tau^{2m+4}),$$

where, e.g.,

$$S^{(0)} = \frac{1}{4} \sum_{b=1}^{n} \frac{(k_b \cdot (k_{n+1} - k_{n+2}))^2}{k_b \cdot (k_{n+1} + k_{n+2})}.$$

[Cachazo, He, Yuan, ‘15]

Examples of double soft theorems

$$A_n = \left(\begin{array}{c:cc} A_{n-2} & \vdots & \vdots \\ \hdashline \cdots & 0 & \frac{\tau^2\, p\cdot q}{-\xi} \\ \cdots & \frac{\tau^2\, p\cdot q}{\xi} & 0 \end{array}\right) \quad \Rightarrow \quad \mathrm{Pf}' A_n = \frac{\tau\, p\cdot q}{\xi_1}\, \mathrm{Pf}' A_{n-2} + \mathcal{O}(\tau^2).$$

For sGal, DBI, EMS (with $m = 1, 0, -1$)

$$M_{n+2} = (k_{n+1} \cdot k_{n+2})^m (\mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \mathcal{S}^{(2)}) M_n + \mathcal{O}(\tau^{2m+4}),$$

where, e.g.,

$$S^{(0)} = \frac{1}{4} \sum_{b=1}^{n} \frac{(k_b \cdot (k_{n+1} - k_{n+2}))^2}{k_b \cdot (k_{n+1} + k_{n+2})}.$$

[Cachazo, He, Yuan, '15]

Such analysis in the CHY setup can also be extended to the study of multiple soft behavior, e.g., [Chakrabarti et al, '17].

Summary

- We reviewed the CHY formalism introduced in recently years for the scattering amplitudes of massless particles at tree level.
- At the core of this formalism is a punctured Riemann sphere, whose configuration is in correspondence to the kinematics via scattering equations.
- Upon this punctured sphere the "amplitude" (i.e., CHY integrand) turn out to have compact closed expressions.
- In this context, various soft theorems naturally arise from the residue theorems associated to the puncture variables for the soft particles. The closed formulas then helps easily determine the soft operators, both at the leading and subleading orders.
- With this method, new soft theorems were also discovered for the double soft limit.

Thank you very much!