Hyperbolic Mass and Gluings of Initial Data

Progress on gravitational physics: 45 years of Belgian-Chilean collaboration

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based on joint work with P. T. Chruściel and E. Delay
arXiv:2112.00095 [math.DG]

Motivation

Why $\Lambda < 0$?

- Anti-de Sitter spacetimes are ubiquitous in nowadays theoretical physics
- Surge of attention due to AdS/CFT correspondence
- Conjecture: Quantum gravity on asymptotically AdS spacetimes in d dimensions is dual to a conformal field theory in $d-1$ dimensions

Motivation

Why $\Lambda < 0$?

- Many *classical* properties of general relativity with $\Lambda < 0$ deserve attention
- One of them is the mass
  - It is only known in special cases whether the mass is bounded from below

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P. T. Chruściel, E. Delay, RW arXiv:2112.00095 [math.DG]:

- proof of existence of certain vacuum initial data sets for GR with $\Lambda < 0$ and negative mass
- previously unknown

Outline

1. Theorem and Motivation
2. Solutions of Interest
3. Mass
4. Sketch of the Proof

1. Theorem and Motivation

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Theorem (P. T. Chruściel, E. Delay, RW)

There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) without boundary at finite distance with scalar curvature

$$R(g) = -6$$

with connected conformal boundary at infinity of arbitrarily high genus and negative total mass.

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There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) without boundary at finite distance with scalar curvature

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- Metric approaches a hyperbolic metric at large distances
- No interior boundary, only conformal boundary at infinity
- Time-symmetric ($K_{ij} = 0$) vacuum initial data with negative cosmological constant

1. Theorem and Motivation

- Hyperbolic space appears as constant time slice of Anti-de Sitter

$$g_{3+1} = -(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2(d\theta^2 + \sin\theta^2 d\phi^2)$$

- Statement about initial data sets for asymptotically locally AdS spacetimes
- Theorem provides better understanding of positivity of mass for asymptotically locally hyperbolic spaces
  - If asymptotically locally AdS spacetime contains spacelike hypersurface that satisfies requirements needed for theorem, theorems carry over to statements about AdS spacetime
- Potential use of bounds in AdS/CFT

2. Solutions of Interest

2. Birmingham-Kottler metrics

Static Solutions of the Vacuum Einstein Equations with $\Lambda < 0$

$$g_{3+1} = -V^2(r)dt^2 + \frac{1}{V^2(r)}dr^2 + r^2 h_k\,, \quad V^2(r) = r^2 + k - \frac{2m_c}{r}$$

where h_k is a t- and r-independent Einstein metric on a 2-dimensional, orientable compact manifold with

$$R(h_k) = 2k\,, \quad k = \{-1, 0, 1\}$$

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- $R(h_1) = 2:$ Riemann sphere with $h_1 = d\theta^2 + \sin\theta^2 d\phi^2$
- $R(h_0) = 0:$ flat torus with $h_0 = d\theta^2 + d\phi^2$
- $R(h_{-1}) = -2:$ higher-genus surface $h_{-1} = d\theta^2 + \sinh\theta^2 d\phi^2$

Remark: BK metrics also referred to as Schwarzschild-AdS

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- $m_c \neq 0$ are nakedly singular unless $V(r_0) = 0$ for some $r_0 > 0$, if $V(r)$ has positive zero $\to$ black hole solutions
- $m_c = 0$, $k = 1$ global AdS spacetime, $t = \mathrm{const.}$ global hyperbolic space
- $m_c = 0$: locally AdS spacetime, $t = \mathrm{const.}$ locally hyperbolic space

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- In space-dimension 3 asymptotically BK equivalent to asymptotically locally hyperbolic
- Mass $E \propto m_c$, measured relativ to $\bar{g} = g(m_c = 0)$

2. Birmingham-Kottler to Horowitz-Myers

Static Solutions of the Vacuum Einstein Equations with $\Lambda < 0$

- Consider a toroidal Birmingham-Kottler metric

$$g_{3+1} = -V^2_{k=0}(r)dt^2 + \frac{1}{V^2_{k=0}(r)}dr^2 + r^2(d\theta^2 + d\psi^2)\,,$$

$$V^2_{k=0}(r) = r^2 - \frac{2m_c}{r}$$

- Wick rotate $t \to i\theta$, $\theta \to it$

$$g_{3+1} = +V^2_{HM}(r)d\theta^2 + \frac{1}{V^2_{HM}(r)}dr^2 + r^2(-dt^2 + d\psi^2)\,,$$

$$V^2_{HM}(r) = r^2 - \frac{2m_c}{r}$$

$\Rightarrow$ Novel solution to vacuum Einstein equations as noticed by Horowitz and Myers

2. Horowitz–Myers Metric

Static Solution of the Vacuum Einstein's Equations with $\Lambda < 0$

$$g_{3+1} = V_{HM}^2(r)d\theta^2 + \frac{1}{V_{HM}^2(r)}dr^2 + r^2(-dt^2 + d\psi^2)$$

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$$V_{HM}^2(r) = r^2 - \frac{2m_c}{r}$$

- For $m_c > 0$, function $V_{HM}(r)$ vanishes at $r = r_0 = (2m_c)^{1/3}$
- Choose period of θ such that no conical singularity at $r = r_0$

$$\theta = \frac{2}{3(m_c)^{1/3}}\phi$$

  where ϕ is 2π-periodic
- Period of θ depends on m_c $\rightarrow$ conformal infinity changes if m_c changes

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- Mass $E \propto -m_c$ when measured with respect to toroidal BK metric with $m_c = 0$

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- Mass $E \propto -m_c$ when measured with respect to toroidal BK metric with $m_c = 0$
- Conjecture (1998): Horowitz-Myers metric minimizes energy if you prescribe conformal structure at infinity
- Conjecture due to AdS/CFT considerations

Initial Data

- For the rest of the talk we work on the level of initial data (M, g_{ij}, K_{ij}) for GR satisfying vacuum constraint equations

$$R(g) = 2\Lambda + K_{ij}K^{ij} - (K^i{}_i)^2$$
$$D_j K^j{}_i - D_i K^j{}_j = 0$$

- In the following, consider time-symmetric case $K_{ij} = 0$, then

$$R(g) = 2\Lambda = -n(n-1)$$

- For $n = 3$

$$R(g) = -6$$

3. Mass

3. Mass

Definition

- Often done using space-time methods
- Also possible using initial data: if g approaches Birmingham-Kottler metric with $m_c = 0$ [1]

$$E = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{r=\tilde{R}} D^j(V) \left(R^i{}_j - \frac{R}{3}\delta^i{}_j \right) dS_i$$

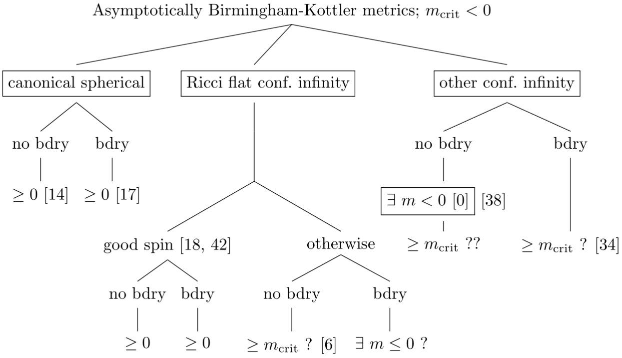
  - $R^i{}_j$ Ricci tensor of g
  - g is the spatial part of the metric
  - Background enters through function V

$$V = \sqrt{r^2 + k}\,, \qquad k \in \{-1, 0, 1\}$$

[1] $dS_i = \sqrt{\det g}\, \partial_i \rfloor dr \wedge d\theta \wedge d\psi$

3. Mass

Positivity of Mass for Negative Cosmological Constant



- Negative mass solutions in toroidal case: Horowitz-Myers metric

4. Sketch of the Proof

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Reminder of Theorem

Theorem (P. T. Chruściel, E. Delay, RW)

There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) without boundary at finite distance with scalar curvature

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Idea:

- Glue together two HM initial data sets at infinity
- Each initial data set has negative mass
- Expectation: gluing results in initial data set with negative mass

4. Sketch of the Proof

Idea: glue together two HM metrics at infinity

Theorem (Isenberg, Lee & Stavrov 2010,)

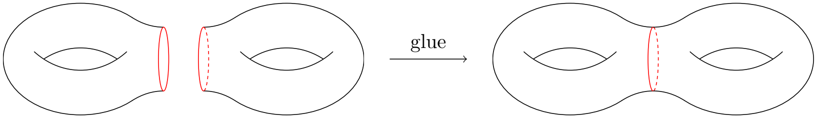
Given two asymptotically locally hyperbolic manifolds with constant scalar curvature (or general relativistic vacuum initial data sets) one can construct a new one by making a connected sum at the conformal boundary at infinity.

4. Sketch of the Proof

Idea: glue together two HM metrics at infinity

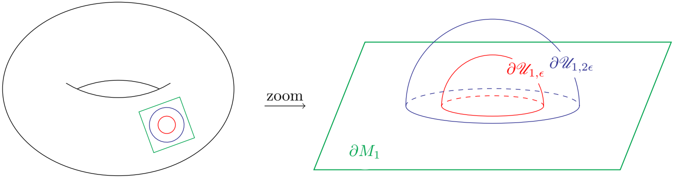
Theorem (Isenberg, Lee & Stavrov 2010, Chruściel, Delay 2015)

Given two asymptotically locally hyperbolic manifolds with constant scalar curvature (or general relativistic vacuum initial data sets) one can construct a new one by making a connected sum at the conformal boundary at infinity. The construction can be localized.

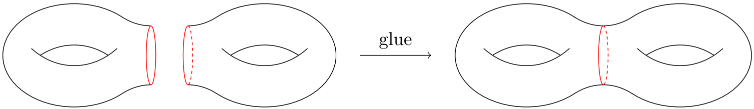


4. Sketch of the Proof

Idea: glue together two HM metrics at infinity

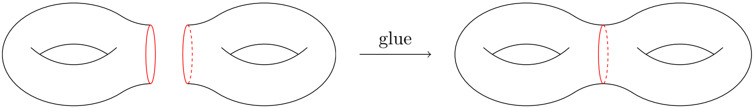


- Metric is exactly hyperbolic inside red half-ball
- Outside blue half-ball metric is exactly what it was before (e.g. Horowitz-Myers in our case)
- Hyperbolic metric can be smoothly extended



4. Sketch of the Proof

How does the Mass change?



- Initial mass defined with respect to locally hyperbolic space with toroidal conformal infinity, final mass defined with respect to locally hyperbolic space with genus-2 conformal infinity

4. Sketch of the Proof

How does the mass change?

- Initial background: locally hyperbolic space with toroidal conformal infinity

$$b = \frac{dr^2}{r^2} + r^2 \underbrace{\left(d\theta^2 + d\varphi^2\right)}_{h_0}$$

- Final genus-2 background: locally hyperbolic space with genus-2 conformal infinity

$$\bar{b} = \frac{d\bar{r}^2}{\bar{r}^2 - 1} + \bar{r}^2 \underbrace{\left(d\bar{\theta}^2 + \sinh^2(\bar{\theta}) d\bar{\varphi}^2\right)}_{h_{-1}}$$

- On each half $h_{-1} = e^{\omega} h_0$
- Inital mass is defined with respect to b, final mass is defined with respect to $\bar{b}$
- One can show that $\bar{r} = e^{-\frac{\omega}{2}} r + \mathit{subleading}$

4. Sketch of the Proof

How does the mass change?

- A few slides before we had

$$E_{generic} = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{r=\tilde{R}} D^j(V) \left(R^i{}_j - \frac{R}{3}\delta^i{}_j \right) dS_i$$

with

$$V = \sqrt{r^2 + k}\,, \qquad k \in \{-1, 0, 1\}$$

- Mass of the initial torus

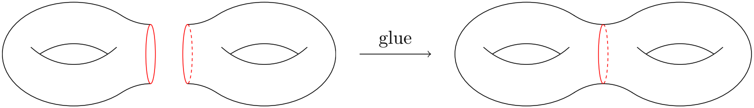
$$E = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{r=\tilde{R}} D^j(r) \left(R^i{}_j - \frac{R}{3}\delta^i{}_j \right) dS_i$$

- Mass of each half of the glued manifold

$$E = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{\bar{r}=\tilde{R}} D^j(\sqrt{\bar{r}^2 - 1}) \left(R^i{}_j - \frac{R}{3}\delta^i{}_j \right) dS_i$$
$$= -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{r=\tilde{R}} D^j(e^{-\omega/2} r) \left(R^i{}_j - \frac{R}{3}\delta^i{}_j \right) dS_i$$

4. Sketch of the Proof

Gluing tori and controlling the mass



- Mass on each half of the manifold depends upon the gluing region
- Sign of the final mass a priori unclear as both the metric and the conformal factor ω depend on the gluing region ϵ

$$E = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{\{r=\tilde{R}\}\times T^2 \setminus D(p,\epsilon)} D^j(e^{-\omega/2} r) \left(R^i{}_j - \frac{R}{3} \delta^i{}_j \right) dS_i$$

- ϵ small needed

4. Sketch of the Proof

Taking the limit $\epsilon \to 0$

Theorem (P. T. Chruściel, E. Delay, RW)

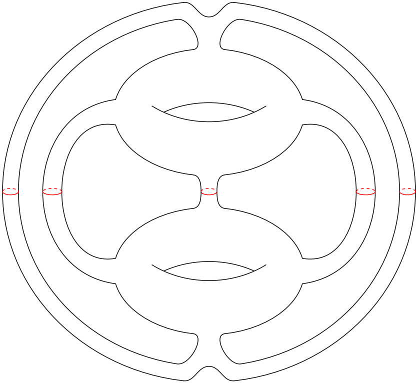
Upon gluing two Horowitz-Myers metrics with coordinate mass m_c, $e^{\omega} \to e^{\omega_0}$ of a punctured torus as $\epsilon \to 0$ with

$$E = -\frac{1}{8\pi} m_c \int_{T^2} e^{-\omega_0/2} d\mu_{h_0}$$

- It follows that if ϵ is chosen small enough, gluing of two Horowitz-Myers metrics gives genus-2 metrics with negative mass

4. Sketch of the Proof

- One obtains higher-genus metrics with negative mass by iterating the construction



Summary

Theorem (P. T. Chruściel, E. Delay, RW)

There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) without boundary at finite distance with scalar curvature

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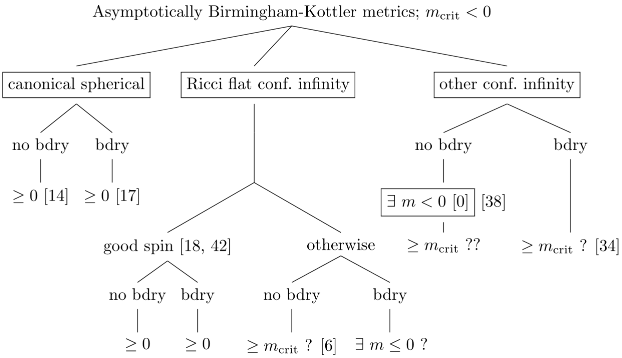
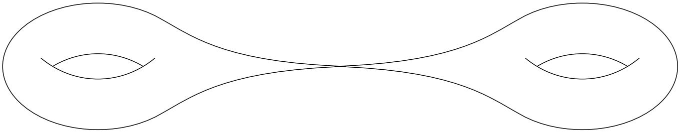
Summary

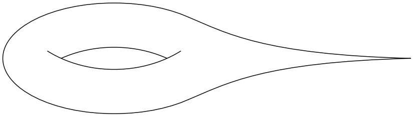

Asymptotically Birmingham-Kottler metrics; $m_{crit} < 0$
canonical spherical
Ricci flat conf. infinity
other conf. infinity
no bdry
bdry
≥ 0 [14]
≥ 0 [17]
good spin [18, 42]
otherwise
no bdry
bdry
≥ 0
≥ 0
no bdry
bdry
$\geq m_{crit}$? [6]
$\exists\ m \leq 0$?
no bdry
bdry
$\exists\ m < 0$ [0] [38]
$\geq m_{crit}$??
$\geq m_{crit}$? [34]

What happens to geometry in limit $\epsilon \to 0$

- Necks become thinner and longer as $\epsilon \to 0$



- As $\epsilon \to 0$ tori seperate: two punctured tori



Topological instability?

- Use construction to **lower** the total mass of a asymptotically locally hyperbolic manifold by a localized deformation near the conformal boundary at infinity
- This is at the cost of **changing the topology** at conformal infinity
- Possible instability? $\rightarrow$ needs further investigation

Thank You!