The 2D Quantum Gravity Partition Function

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Workshop on:

"Topics in gravitational physics: from black holes to asymptotic symmetries"
"Progress on gravitational physics: 45 years of Belgian-Chilean collaboration"

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Liouville Field Theory

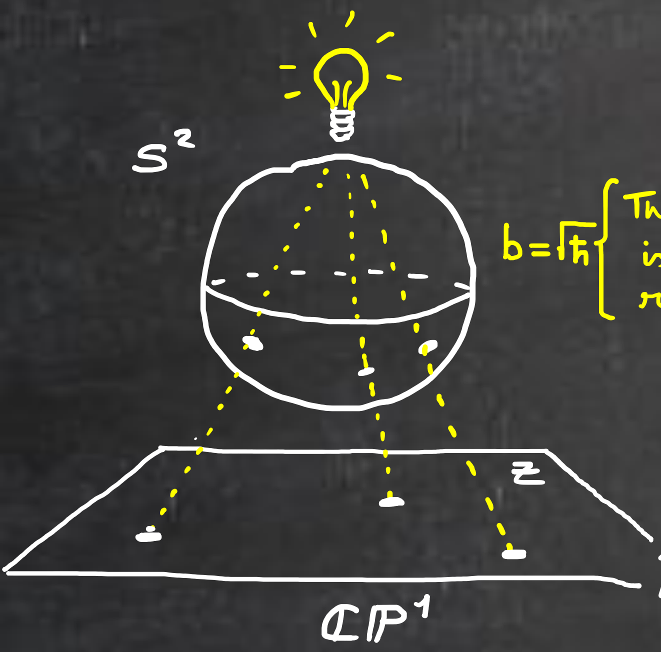
- String theory (Polyakov 1981)
- 2D Quantum gravity (Distler & Kawai 1989)
- 3D Gravity in AdS (Coussaert, van Driel, Henneaux 1995)
- 4D $\mathcal{N}=2$ SCFT's (Alday-Gaiotto-Tachikawa '09)
- Relation to other CFT's, e.g. WZW (Stoyanovsky)
- Relation to Matrix Models (cf. Seiberg 1990)
- Toy model for quantum cosmology (Dabholkar et al.)
- Celestial holography (Taylor et al.)
- [...]

- Uniformization problem (Liouville 1882)
- Conformal mapping
- Statistical physics

Timelike Liouville theory

- Time-dependent backgrounds in string theory
- Black hole physics and singularities

Liouville Field Theory (spacelike)



Liouville field

background charge

Liouville cosmological constant

$$S[\Lambda] = \frac{1}{4\pi}\int d^2z \left[+\partial\phi\bar\partial\phi + \frac{Q}{\sqrt{2}} R\phi + \underbrace{4\pi\Lambda e^{\sqrt{2}b\phi}} \right] , \quad \text{with} \quad Q = b + \frac{1}{b}$$

$\phi \to \phi + \frac{1}{\sqrt{2}b}\log\left(\frac{\Lambda_*}{\Lambda}\right)$

$b=\sqrt{\hbar}$ $\Bigg\{$ The semiclassical limit is $b \to 0$. To see this, rescale $\phi \to \phi/b$

2D scalar curvature

There exists another marginal operator:

$$4\pi\tilde\Lambda \int e^{\sqrt{2}\frac{1}{b}\phi} d^2z$$

which is "dual" under $b \leftrightarrow \frac{1}{b}$

Central charge $\quad C = 1 + 6Q^2 \geq 25$

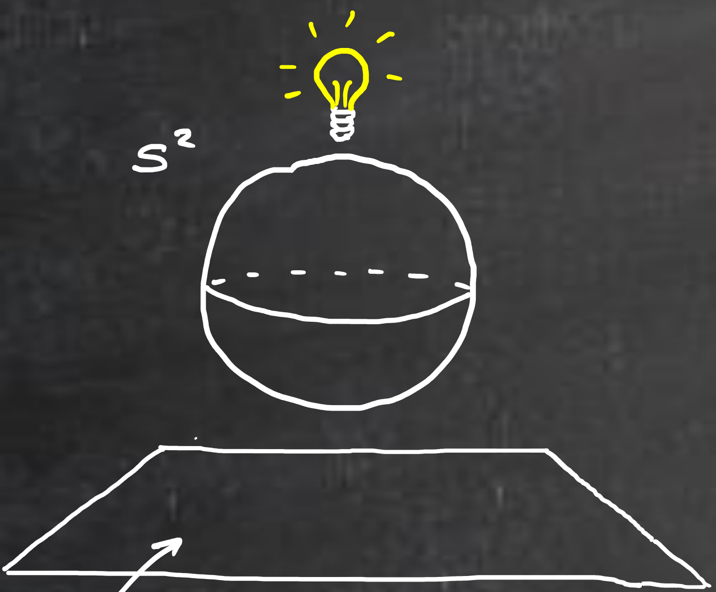
Primary operators $\quad V_\alpha(z) = :e^{\sqrt{2}\alpha\phi(z)}:$

with $\alpha \in \frac{Q}{2} + i\mathbb{R}$, $\Delta_\alpha = \alpha(Q-\alpha)$

Correlation functions

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) \rangle = \prod_{i<j} |z_i - z_j|^{2\left(\sum_{k=1}^{3}\Delta_{\alpha_k} - 2\Delta_{\alpha_i} - 2\Delta_{\alpha_j}\right)} \underbrace{C(\alpha_1,\alpha_2,\alpha_3)}_{\text{DOZZ formula}}$$

Liouville Field Theory (spacelike)



$\mathbb{C}\mathbb{P}^1$

$$S[\Lambda] = \frac{1}{4\pi}\int d^2z \left[+\partial\phi\,\bar{\partial}\phi + \frac{Q}{\sqrt{2}} R\,\phi + 4\pi\Lambda\, e^{\sqrt{2}b\phi} \right] \quad , \quad \text{with} \quad Q = b + \frac{1}{b}$$

$$Z[\Lambda] = \int_{\phi(\mathbb{C}P^1)} \frac{D\phi \;\; e^{-S[\Lambda]}}{Vol(PSL(2,\mathbb{C}))}$$

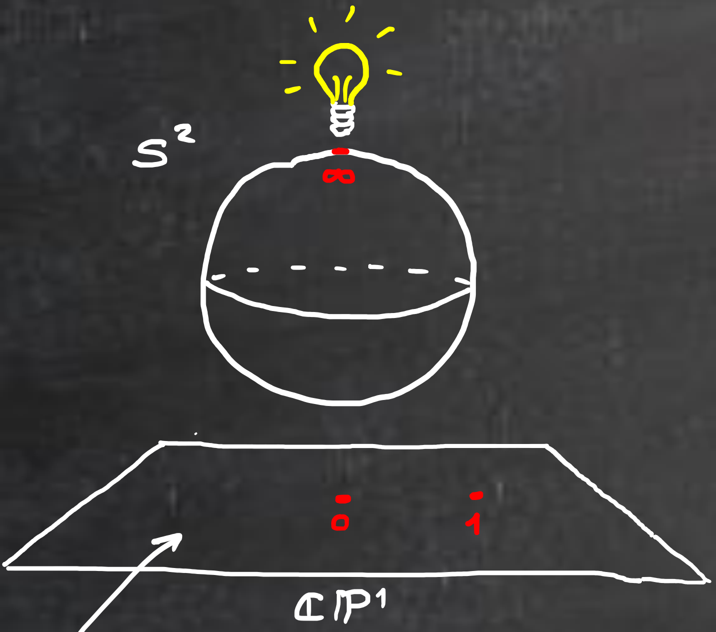
Stabilizer

Möbius group (conf. Killing group)

$$PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathbb{Z}_2$$

$$Vol(PLS(2,\mathbb{C})) = \int_{\mathbb{C}^3} \frac{d^2z_1\, d^2z_2\, d^2z_3}{|z_1 - z_2|^2\, |z_1 - z_3|^2\, |z_2 - z_3|^2}$$

Liouville Field Theory (spacelike)



$$S[\Lambda] = \frac{1}{4\pi}\int d^2z \left[+\partial\phi\,\bar{\partial}\phi + \frac{Q}{\sqrt{2}} R\,\phi + 4\pi\Lambda\, e^{\sqrt{2}b\phi} \right] , \quad \text{with} \quad Q = b + \frac{1}{b}$$

$$\frac{d^3}{d\Lambda^3} Z[\Lambda] = \int_{\phi(\mathbb{CP}^1)} \frac{D\phi \quad e^{-S[\Lambda]}}{\cancel{Vol(PSL(2,\mathbb{C}))}} \cancel{\int_{\mathbb{C}^3} d^2z_1\, d^2z_2\, d^2z_3} \; e^{\sqrt{2}b\phi(z_1 = 0)}\, e^{\sqrt{2}b\phi(z_2 = 1)}\, e^{\sqrt{2}b\phi(z_3 = \infty)}$$

Stabilizer

$$\frac{d^3}{d\Lambda^3} Z[\Lambda] = \left\langle e^{\sqrt{2}b\phi(0)}\, e^{\sqrt{2}b\phi(1)}\, e^{\sqrt{2}b\phi(\infty)} \right\rangle = C(b,b,b)$$

Known expression (DOZZ)

Möbius group (conf. Killing group)

$$PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathbb{Z}_2$$

$$Vol(PLS(2,\mathbb{C})) = \int_{\mathbb{C}^3} \frac{d^2z_1\, d^2z_2\, d^2z_3}{|z_1 - z_2|^2\, |z_1 - z_3|^2\, |z_2 - z_3|^2}$$

$z_1 = 0,\ z_2 = 1,\ z_3 = \infty$

Spacelike Liouville theory partition function on S^2

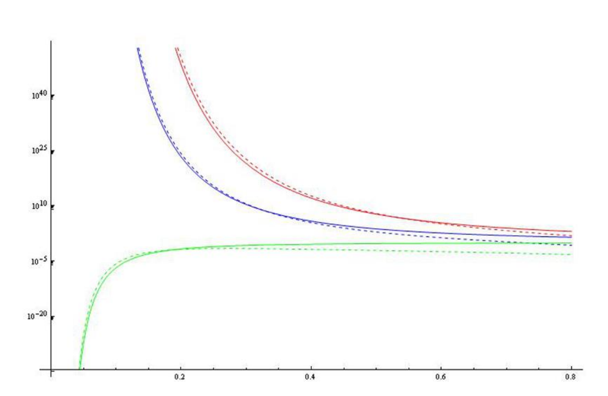
$$Z[\Lambda] = \frac{(1-b^2)}{\pi^3(b+b^{-1})}\left(\pi\Lambda\,\frac{\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{1-b^{-2}}\frac{\Gamma(1-b^2)\,\Gamma(1-b^{-2})}{\Gamma(b^2)\;\Gamma(b^{-2})}$$

- It is non-zero, $\frac{\infty}{\infty}$
- Analytic structure governed by the Γ-functions
- Non-symmetric under $b \leftrightarrow 1/b$,

> Notice that this expression in *not* self-dual, unlike the three-point function itself. This fact looks somewhat surprising in the Liouville context and likely needs better understanding.

Al. Zamolodchikov 2005

- Semiclassical limit ($b \to 0$) gives:



$$\hat{Z} \simeq e^{b^{-2}}\left(\frac{\text{Area}}{\pi}\right)^{-b^{-2}} = e^{-S_{class}} = e^{-\frac{1}{16\pi G}\int_{AdS} d^3x\,\left(R+\frac{2}{\ell^2}\right)}$$

$b \simeq \sqrt{\hbar} \simeq 0$

see Krasnov (2001)

with $c = 1+6Q^2 = \frac{3\ell}{2G}$

Brown & Henneaux (1986)

Timelike Liouville Field Theory

$$S[\Lambda] = \frac{1}{4\pi}\int d^2z\left[-\partial\phi\bar{\partial}\phi + \frac{Q}{\sqrt{2}}R\phi + 4\pi\Lambda e^{\sqrt{2}b\phi}\right] \quad , \quad \text{with} \quad Q = b - \frac{1}{b}$$

$$c = 1 - 6Q^2 \leq 1$$

Timelike Liouville Field Theory

The "dual" marginal operator
$4\pi\tilde{\Lambda}\, e^{-\sqrt{2}\phi/b}$

$$S[\Lambda] = \frac{1}{4\pi}\int d^2z \left[-\partial\phi\bar{\partial}\phi + \frac{Q}{\sqrt{2}} R\phi + 4\pi\Lambda\, e^{\sqrt{2}b\phi}\right], \quad \text{with} \quad Q = b - \frac{1}{b}$$

$$c = 1 - 6Q^2 \leq 1$$

$$Z[\Lambda] = \langle 0|0\rangle_{\text{timelike}} \equiv \int_{\phi(\mathbb{CP}^1)} \frac{\mathcal{D}\phi\; e^{-S[\Lambda]}}{\mathrm{Vol}(PSL(2,\mathbb{C}))}$$

Analytic continuation
$$\begin{cases} \phi \to -i\phi \\ b \to +ib \end{cases}$$

$$\frac{d^3}{d\Lambda^3} Z[\Lambda] = \underbrace{\langle 0|\, e^{\sqrt{2}b\phi(0)}\, e^{\sqrt{2}b\phi(1)}\, e^{\sqrt{2}\phi(\infty)}\,|0\rangle}_{C(b,b,b)\,=\,0\ ???} = \int \cancel{\prod_{i=1}^{3} d^2z_i} \int \frac{\mathcal{D}\phi\; e^{-S[\Lambda]}}{\mathrm{Vol}(\cancel{PSL(2,\mathbb{C})})} \prod_{\ell=1}^{3} e^{\sqrt{2}b\phi(z_\ell)}$$

Timelike DOZZ formula

- Strominger & Takayanagi (2003)
- Schomerus (2003)
- Kostov & Petkova (2006)
- Zamolodchikov (2005)
- Harlow-Maltz-Witten (2011)
- G.G. (2011)
- Ribault & Santachiara (2015)

[...]

We want to compute $Z[\Lambda]$ for the Timelike theory directly:

$$\phi = \tilde{\phi} + \phi_0 \,, \quad \int D\phi = \int D\tilde{\phi} \int_{\mathbb{R}} d\phi_0$$
$$\langle \phi \rangle = \phi_0$$

$$Z[\Lambda] = \int_{\phi(\mathbb{CP}^1)} \mathcal{D}\phi \, \frac{e^{-S[\Lambda]}}{\mathrm{Vol}(PSL(2,\mathbb{C}))} = \int_{\phi(\mathbb{CP}^1)} \mathcal{D}\phi \, \frac{e^{+\frac{1}{4\pi}\int(\partial\phi)^2 - \frac{Q}{4\pi\sqrt{2}}\int R\phi}}{\mathrm{Vol}(PSL(2,\mathbb{C}))} \, e^{-\Lambda\int e^{\sqrt{2}b\phi}} =$$

$$= \int \mathcal{D}\tilde{\phi} \, \frac{e^{\frac{1}{4\pi}\int(\partial\tilde{\phi})^2 - \frac{Q}{4\pi\sqrt{2}}\int R\tilde{\phi}}}{\mathrm{Vol}(PSL(2,\mathbb{C}))} \int_{\mathbb{R}} d\phi_0 \, e^{-\frac{Q}{\sqrt{2}}\phi_0 \overset{2}{\chi(S^2)}} \int_{\mathbb{R}_+} d\eta \, e^{-\Lambda\eta} \, \delta\!\left(\eta - e^{\sqrt{2}b\phi_0}\int d^2z \, e^{\sqrt{2}b\tilde{\phi}}\right) =$$

$$= \frac{1}{b}\int \mathcal{D}\tilde{\phi} \, \frac{e^{-S[\Lambda=0]}}{\mathrm{Vol}(PSL(2,\mathbb{C}))} \left(\int_{\mathbb{C}} d^2z \, e^{\sqrt{2}b\phi}\right)^{\frac{Q}{b}} \int_{\mathbb{R}_+} d\eta \, \eta^{-1-\frac{Q}{b}} e^{-\Lambda\eta} =$$

$$= \frac{\Lambda^{\frac{Q}{b}}\Gamma(-Q/b)}{b} \int_{\mathbb{C}^{Q/b}} \prod_{n=4}^{Q/b} d^2z_n \int \mathcal{D}\tilde{\phi} \, \frac{e^{-S[\overset{\text{free}}{0}]}}{\cancel{\mathrm{Vol}(PSL(2,\mathbb{C}))}} \prod_{l=1}^{Q/b} e^{\sqrt{2}b\phi(z_l)} = \frac{\Lambda^{\frac{Q}{b}}\Gamma(-Q/b)}{b} \int \prod_{n=4}^{Q/b} d^2z_n \left\langle \prod_{l=1}^{Q/b} e^{\sqrt{2}b\phi(z_l)} \right\rangle_{\text{free}}$$

$z_1 = 0$
$z_2 = 1$
$z_3 = \infty$

$$\left\langle e^{\sqrt{2}b\phi(z_i)} e^{\sqrt{2}b\phi(z_j)} \right\rangle_{\text{free}} = e^{2b^2\langle\phi(z_i)\phi(z_j)\rangle} = |z_i - z_j|^{4b^2}$$

We want to compute $Z[\Lambda]$ for the timelike theory directly:

$$\underbrace{\left\langle e^{\sqrt{2}b\phi(z_1)} \cdots e^{\sqrt{2}b\phi(z_m)} \right\rangle_{\text{free}}}$$

with $m = \frac{Q}{b} - 3 = -2 - b^{-2}$

$$Z[\Lambda] = \int_{\phi(\mathbb{CP}^1)} \frac{\mathcal{D}\phi \; e^{-S[\Lambda]}}{\mathrm{Vol}(PSL(2,\mathbb{C}))} = \frac{\Gamma(-m-3)\,\Lambda^{m+3}}{b} \int_{\mathbb{C}^m} \prod_{n=1}^{m} d^2 z_n \left(\prod_{i=1}^{m} |z_i|^{4b^2} |1-z_i|^{4b^2} \prod_{t=1}^{m} \prod_{l=1}^{t-1} |z_t - z_l|^{4b^2} \right) =$$

Assume for a while $m \in \mathbb{Z}_{>0}$

$$= \frac{\Lambda^{m+3}}{b} \Gamma(-m-3)\Gamma(m+1)\, \pi^m \left(\frac{\Gamma(1+b^2)}{\Gamma(-b^2)} \right)^m \prod_{t=1}^{m} \frac{\Gamma(tb^2)\Gamma(-tb^2)}{\Gamma(1+tb^2)\Gamma(1-tb^2)} \prod_{n=2}^{m+1} \frac{\Gamma^2(1+nb^2)}{\Gamma^2(-nb^2)} = [\cdots] =$$

$$\boxed{Z[\Lambda] = \frac{(1+b^2)}{\pi^3 \left(b - \frac{1}{b}\right)} \left[\frac{\pi \Lambda \Gamma(-b^2)}{\Gamma(1+b^2)} \right]^{1-\frac{1}{b^2}} \frac{\Gamma(1+b^2)\,\Gamma(1+b^{-2})}{\Gamma(-b^2)\,\Gamma(-b^{-2})}}$$

Timelike Liouville partition function on the sphere topology!