The motion of test bodies in Kerr spacetime

a review, including some personal digressions

Solvay Workshop on
"Progress on gravitational physics:
45 years of Belgian-Chilean collaboration"

12th April 2023

Adrien Druart

[ULB, Physique Théorique et Mathématique]

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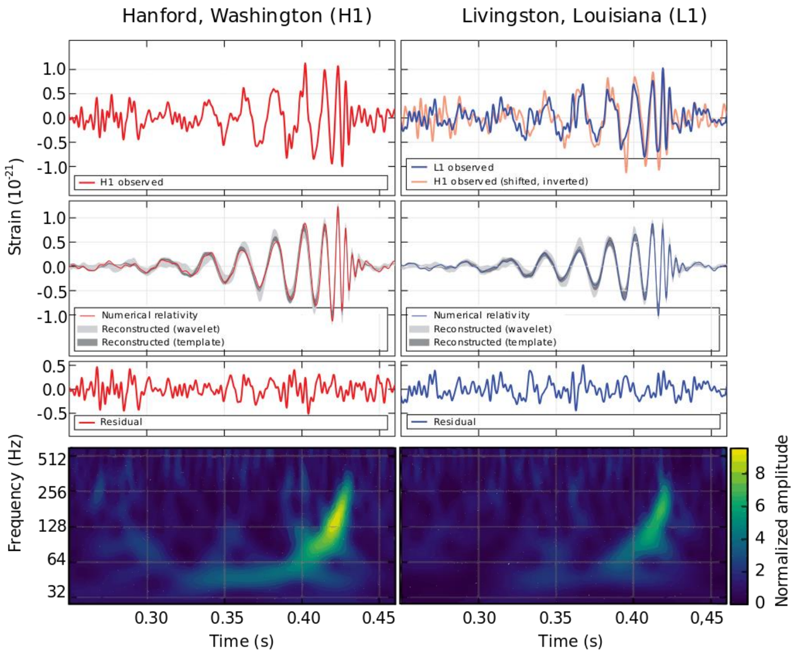
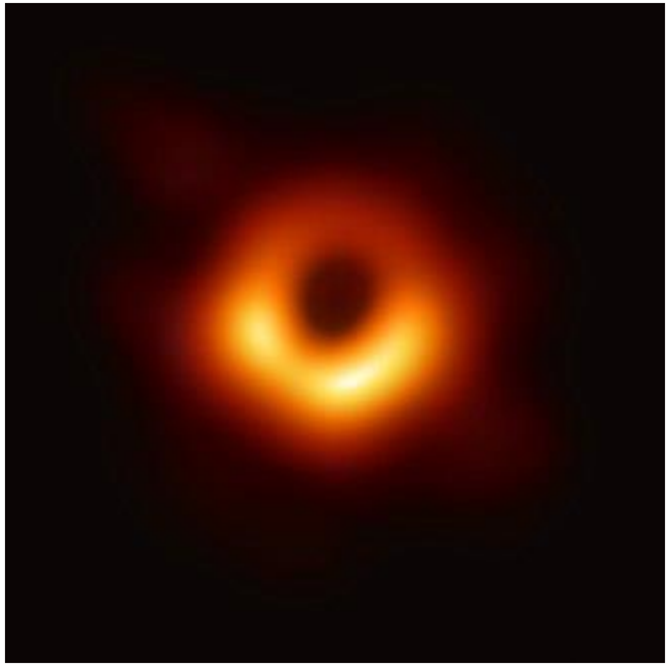
Exciting times for black holes…

- September 2015: GW150914 (first detection of BBH coalescence)
- April 2019: direct image of the shadow of M87* SMBH

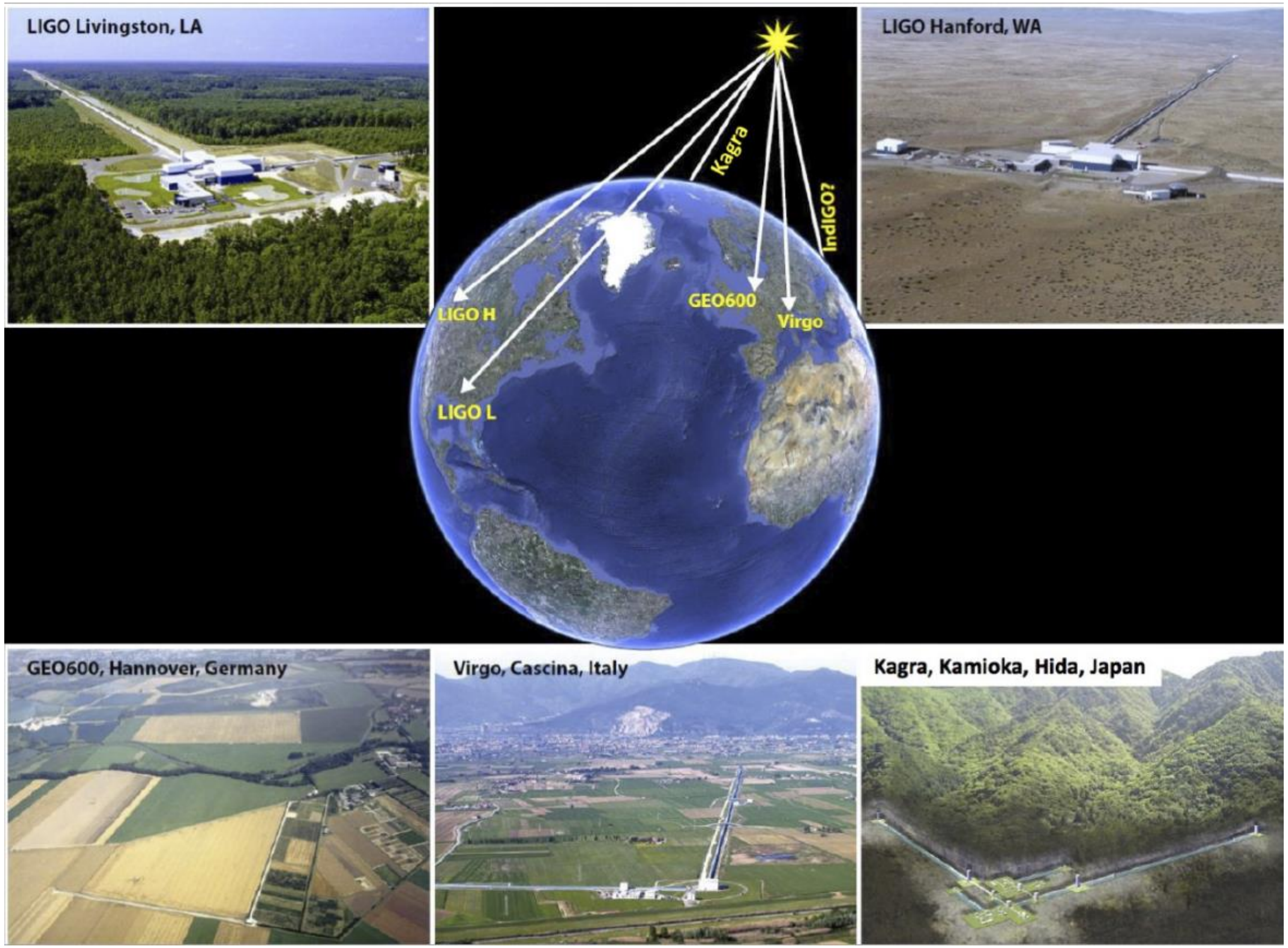


[arXiv:1602.03837]



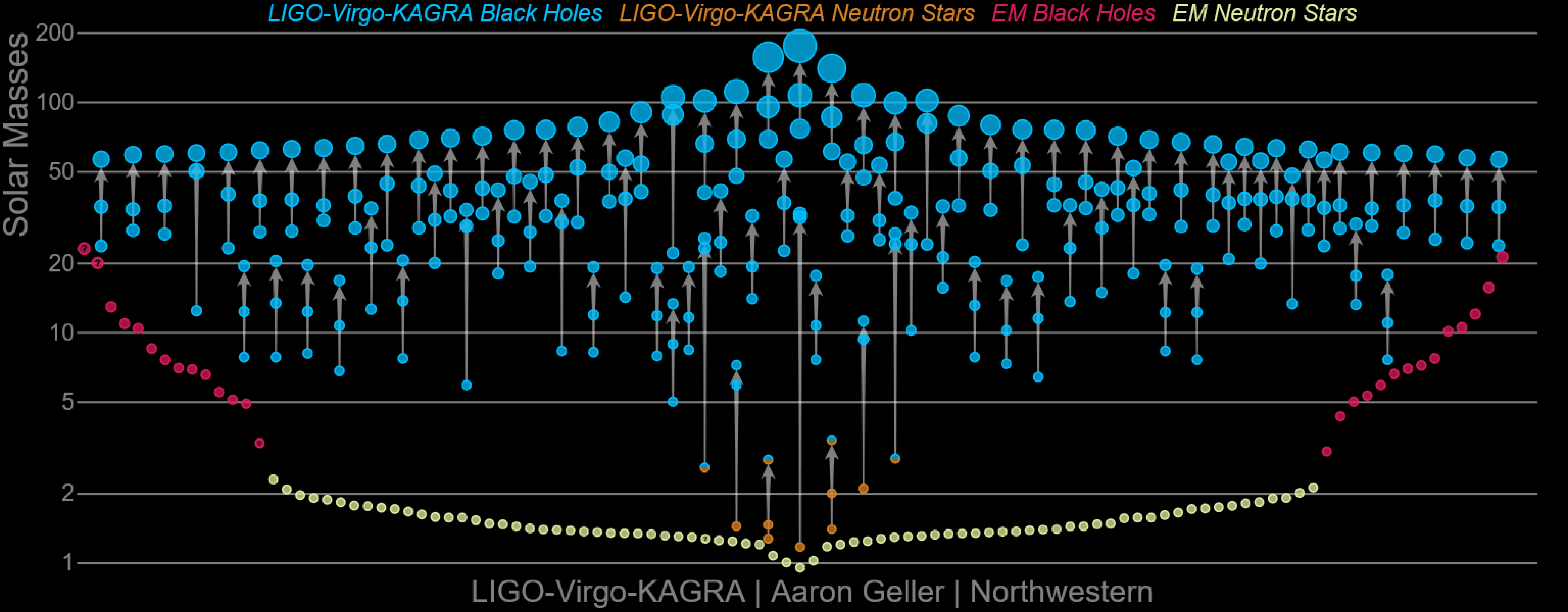
[Image credit: EHT]

Current gravitational wave detectors

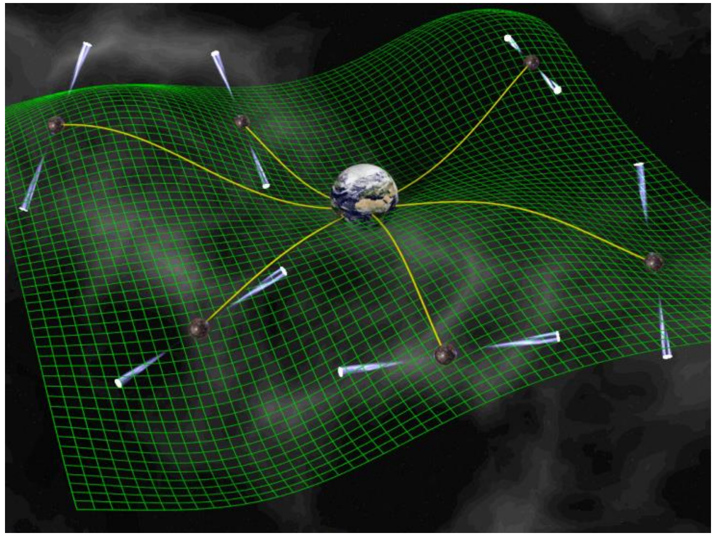


[Image credit: LIGO collaboration]

Masses in the Stellar Graveyard

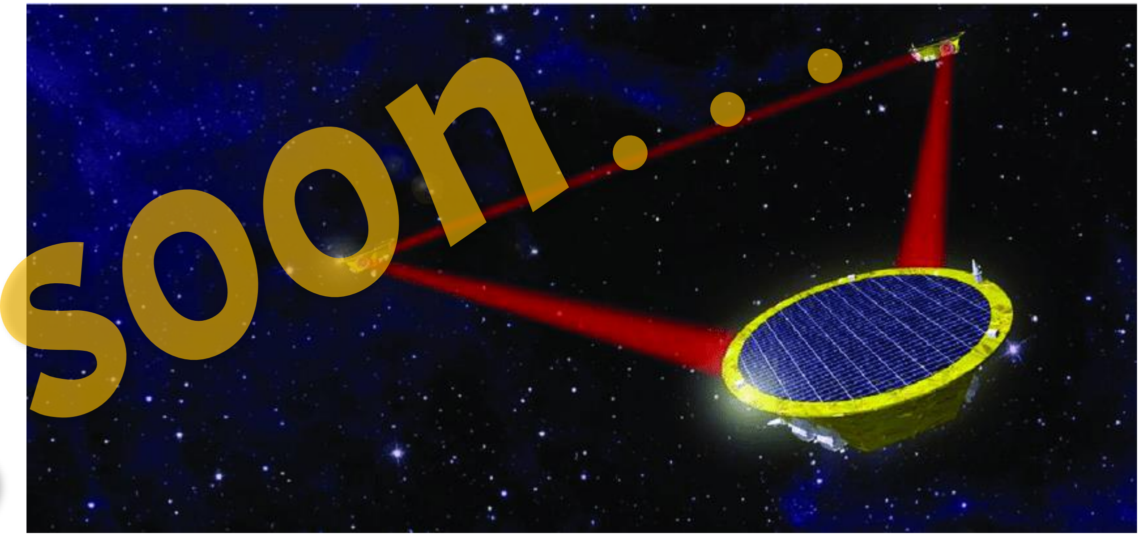


[Image credit: David J. Champion]



Pulsar Timing Array

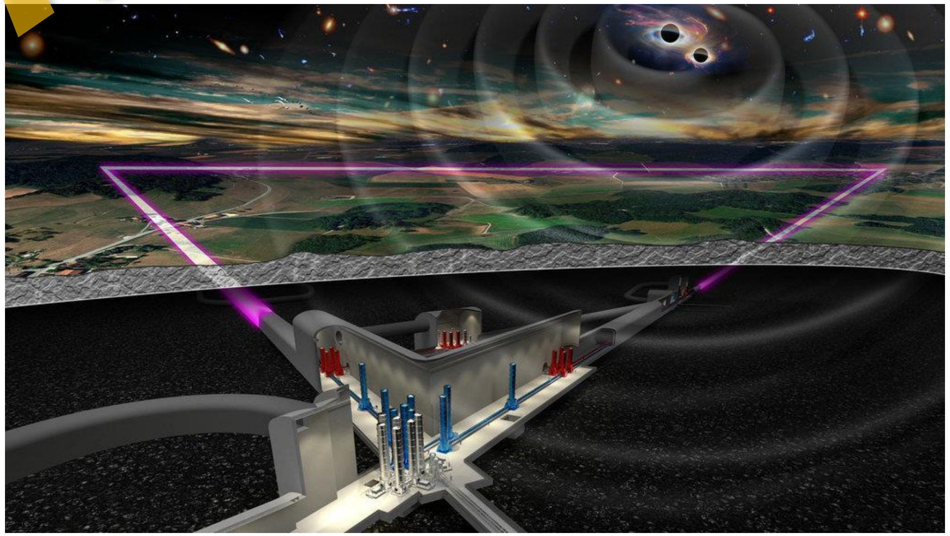
Laser Interferometer Space Antenna (LISA)



[Image credit: LISA consortium]

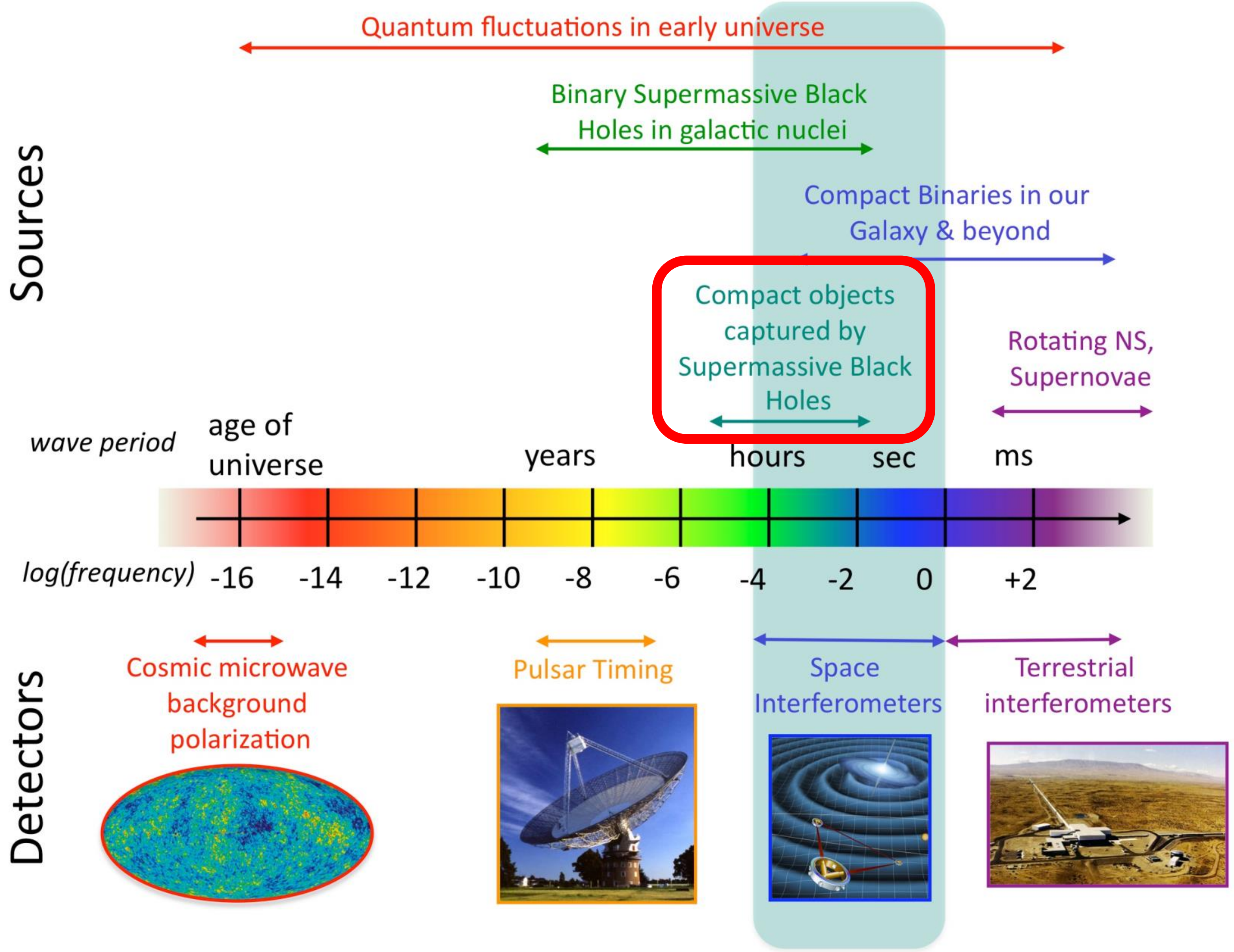
Coming soon…

Einstein Telescope



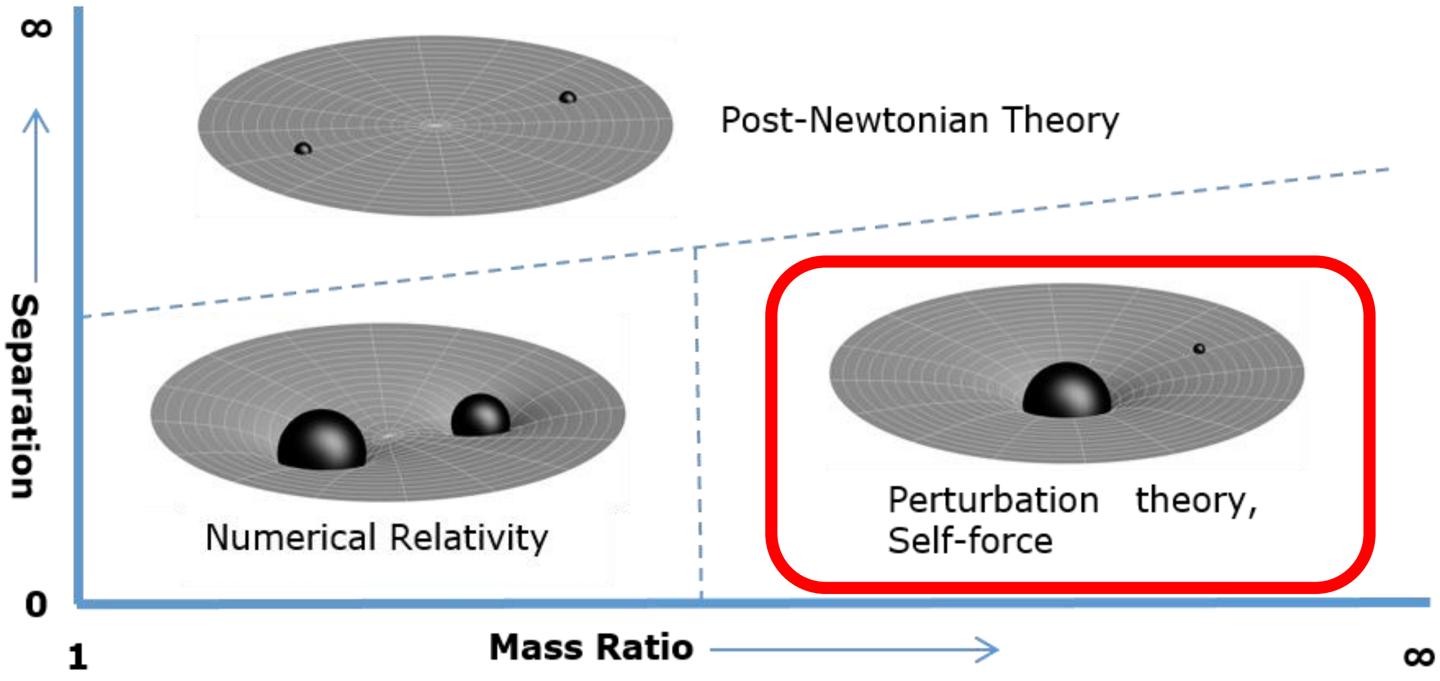
[Image credit: ET collaboration]

The Gravitational Wave Spectrum



[Image credit: NASA Goddard Space Flight Center]

The two body problem in GR



- **Extreme Mass Ratio Inspirals (EMRIs):**

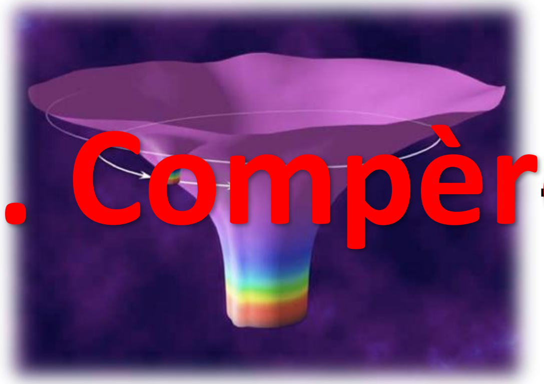
$$10^{-4} \leq \epsilon \triangleq \frac{\mu}{M} \leq 10^{-6}$$

- **Obtain precise waveform models is a community-scaled effort!**

Extreme mass ratio inspirals physics @ ULB

Self force and transition
(L. Küchler, G. Lhost)

G. Compère



[Credit: NASA]

Environemental effects
(L. Machet)

Finite size effects
(A. Druart)

Finite size effects: equation of motion

- Secondary modelled as moving on a single worldline…
- …but is not a point particle => finite size effects (spin, tidal deformability…)
- "Forced geodesic equation" description, finite size effects as perturbation to geodesic motion
- GSF/finite size effects arise at the same order in the small mass ratio expansion

$$\frac{Dz^\mu}{d\tau} = \epsilon\, f^\mu[g_{\alpha\beta}, z^\alpha, T_{\alpha\beta}]$$

$f^\mu = 0$: geodesic motion

$f^\mu = f^\mu_{GSF}$: self-force corrections

$f^\mu = f^\mu_{MP}$: finite-size effects

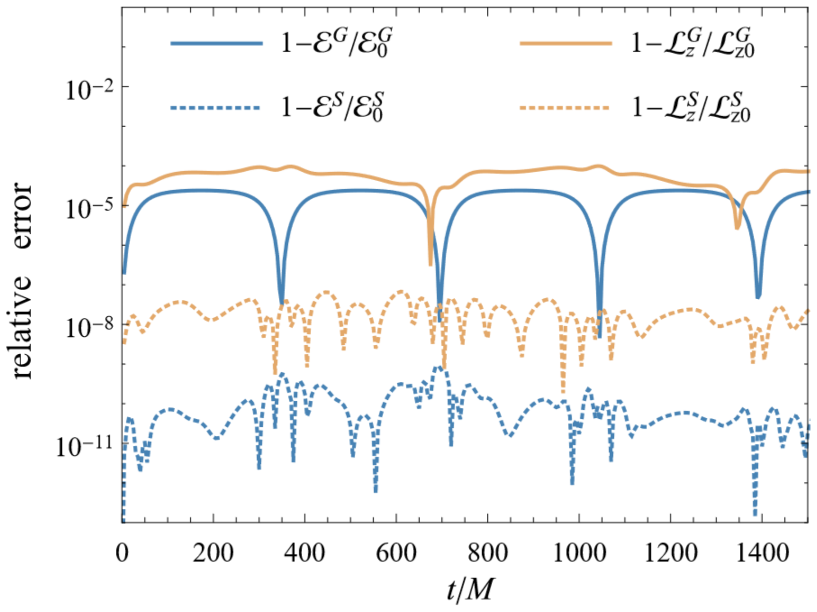
Finite size effects: observational motivations

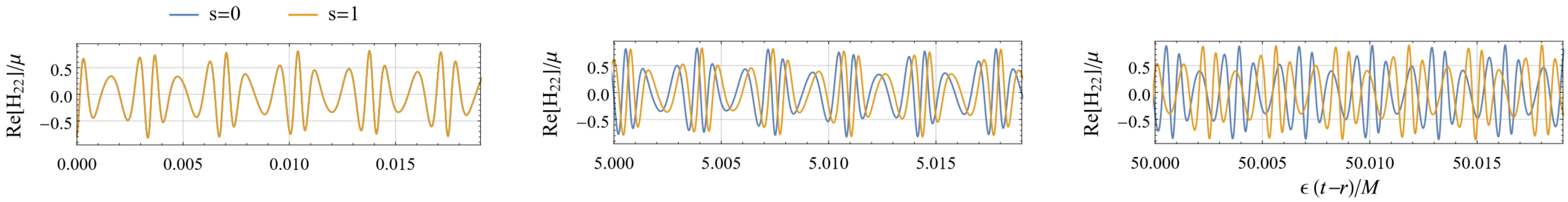
- **LISA parameter extraction requires**

$$\Delta\phi \sim 1 \text{ rad}$$

- **Orbital phase** [Flanagan and Hinderer 2008]

$$
\begin{aligned}
\phi = {}& \phi^{(1)}_{\rm avg} && \mathcal{O}(\epsilon^{-1}) \\
& + \phi^{(1)}_{\rm osc} + \phi^{(2)}_{\rm avg} + \phi^{(1)}_{\rm spin} && \mathcal{O}(\epsilon^{0}) \\
& + \phi^{(2)}_{\rm osc} + \phi^{(3)}_{\rm avg} + \phi^{(2)}_{\rm spin} && \mathcal{O}(\epsilon^{1})
\end{aligned}
$$





[Warburton et al. 2017]

Plan of the talk

$$\frac{Dz^{\mu}}{d\tau} = \epsilon\, f^{\mu}[g_{\alpha\beta}, z^{\alpha}, T_{\alpha\beta}]$$

$f^{\mu} = 0$: geodesic motion

$f^{\mu} = f^{\mu}_{GSF}$: self-force corrections

$f^{\mu} = f^{\mu}_{MP}$: finite-size effects

I. Kerr metric and its geodesics

II. Test bodies in curved spacetime

III. Conserved quantities

IV. Hamiltonian formulation

I. Kerr metric and its geodesics

Kerr metric

[Kerr 1963]

- Most generic asymptotically-flat and stationary solution of vacuum Einstein equations

$$\mathrm{d}s^2 = -\frac{\Delta(r)}{\Sigma(r,\cos\theta)}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\right)^2 + \Sigma(r,\cos\theta)\left(\frac{\mathrm{d}r^2}{\Delta(r)} + \mathrm{d}\theta^2\right) + \frac{\sin^2\theta}{\Sigma(r,\cos\theta)}\left[(r^2+a^2)\mathrm{d}\varphi - a\mathrm{d}t\right]^2$$

$$\Delta(r) \triangleq r^2 - 2Mr + a^2, \qquad \Sigma(r,\cos\theta) \triangleq r^2 + a^2\cos^2\theta.$$

- Two free parameters: mass M and spin a, satisfying $a^2 \leq M^2$
- Stationary and axisymmetric: two Killing vectors

$$\xi \triangleq \partial_t \qquad \eta \triangleq \partial_\varphi$$

In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity, discovered by the New Zealand mathematician Roy Kerr, provides the absolute exact representation of untold numbers of massive black holes that populate the universe. This “shuddering before the beautiful,” this incredible fact that a discovery motivated by a search after the beautiful in mathematics should find its exact replica in Nature, persuades me to say that beauty is that to which the human mind responds at its deepest and most profound level.

S. Chandrasekhar

Also impressive (in my opinion) is the possibility of solving explicitly numerous problems in Kerr spacetime !

Geodesic equations in Kerr are separable !

[Carter 1968]

- **Hamiltonian for geodesic motion**

$$H(x^\alpha, p_\alpha) = \frac{1}{2\mu} g^{\mu\nu}(x^\alpha) p_\mu p_\nu$$

- **Hamilton-Jacobi equation separated and solved by B. Carter in 1968**

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = a\big(L_0 - aE_0 \sin^2\theta\big) + (r^2 + a^2)\frac{P_0(r)}{\Delta(r)},$$
$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \pm_r \sqrt{R(r)},$$
$$\frac{\mathrm{d}\cos\theta}{\mathrm{d}\lambda} = \pm_\theta \sqrt{\Theta(\cos^2\theta)},$$
$$\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = -aE_0 + L_0 \csc^2\theta + a\frac{P_0(r)}{\Delta(r)},$$

- **Carter constant**

$$Q_0 = p_\theta^2 + \cos^2\theta\left[a^2(1 - E_0^2) + \left(\frac{L_0}{\sin\theta}\right)^2\right]$$

Constants of motion and integrability (I)

- Mass μ conserved
- Two Killing vectors => two quantities conserved along geodesic motion

$$E_0 \triangleq -p_\mu \xi^\mu, \qquad L_0 \triangleq p_\mu \eta^\mu$$

- Carter constant is related to the existence of a hidden symmetry of Kerr spacetime

$$K_0 \triangleq K_{\mu\nu} p^\mu p^\nu \qquad (Q_0 = K_0 - (L_0 - aE_0)^2)$$

- Rank-2 Killing tensor, originating from a rank-2 Killing-Yano tensor

$$K_{\mu\nu} = Y_\mu{}^\lambda Y_{\nu\lambda}$$

$$\frac{1}{2} Y_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = a\cos\theta \mathrm{d}r \wedge (\mathrm{d}t - a\sin^2\theta \mathrm{d}\varphi) + r\sin\theta \mathrm{d}\theta \wedge \left[(r^2 + a^2)\mathrm{d}\varphi - a\mathrm{d}t\right]$$

Constants of motion and integrability (II)

	STANDARD	CONFORMAL
Killing vector	$\nabla_{(\mu}X_{\nu)} = 0$	$\nabla_{(\mu}X_{\nu)} = \frac{1}{4}g_{\mu\nu}\nabla_\rho X^\rho$
Killing-Yano tensor (antisymmetric)	$\nabla_{(\mu}Y_{\nu)\rho} = 0$	$\nabla_{(\mu}Y_{\nu)\rho} = -2g_{\mu[\nu}\xi_{\rho]}$
Killing tensor (symmetric)	$\nabla_{(\mu}K_{\nu\rho)} = 0$	$\nabla_{(\mu}K_{\nu\rho)} = K_{(\mu}g_{\nu\rho)}$

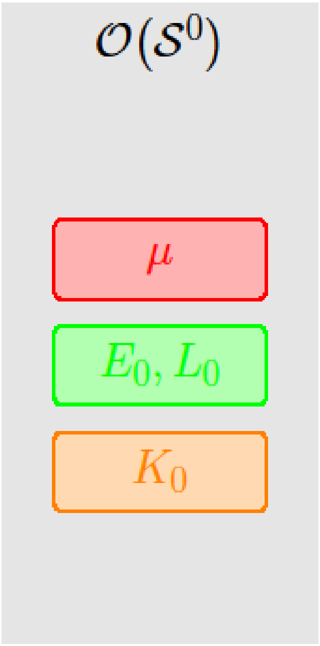
- Killing-Yano tensors more fundamental objects than Killing tensors

  => see later conserved quantities for spinning bodies

- N=4 Hamiltonian system + 4 independent constants of the motion in involution

  => completely integrable

- Explicit solutions + classification of Kerr geodesic obtained



Action-angle variables formulation

[Schmidt 2002, Flanagan and Hinderer 2008]

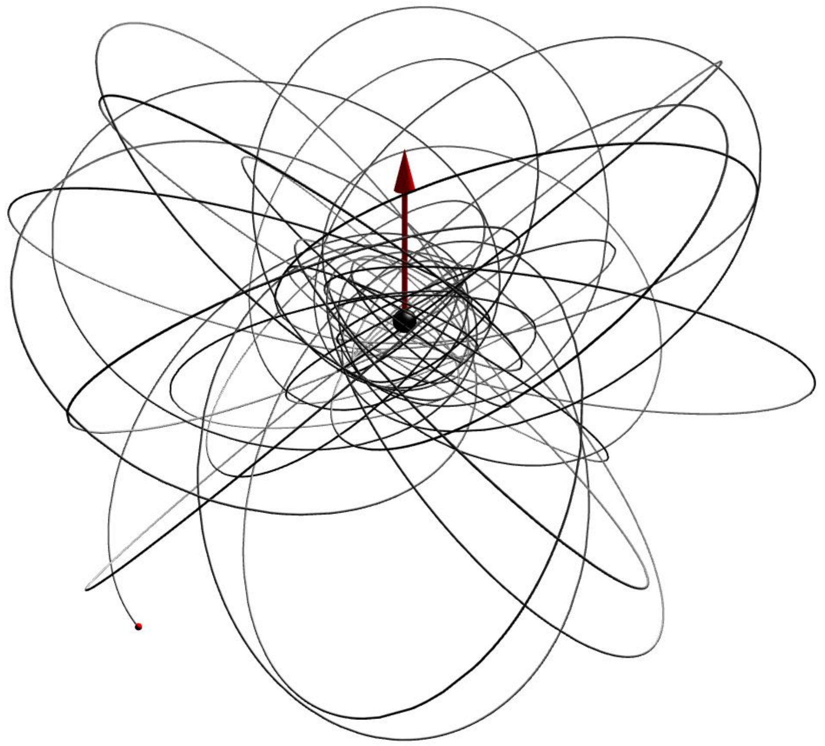
- Kerr (radially) bounded geodesic motion in periodic in its radial, azimutal and polar directions
- Periodicity made explicit using generialized action-angle variables

$$(x^\mu, p_\mu) \to (q^\mu, J_\mu)$$

such that

$$\dot{q}^\mu = \omega^\mu(J), \quad \dot{J}_\mu = 0$$

- Generalizable to self-forced motion [Flanagan and Hinderer 2008]



[Credit: Maarten van de Meent]

Take home message from Part I

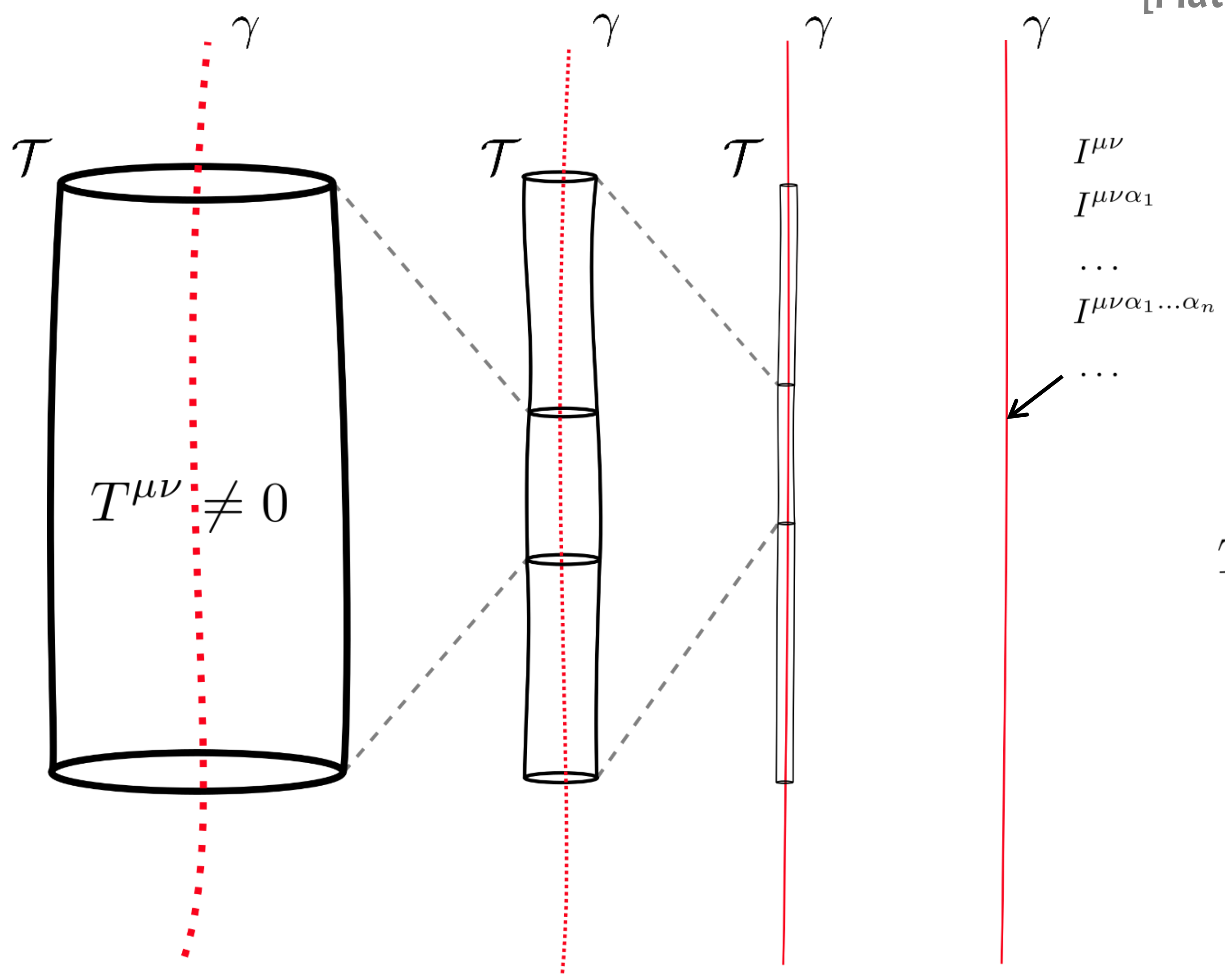
- Kerr (radially) bounded geodesic motion is
  - triperiodic
  - separable
  - integrable

Generalization of these properties if one includes finite size effects ?

II. Test bodies in curved spacetime

Gravitational skeletonization

[Mathisson 1937, Papapetrou 1951, Tulczyjew 1959, Dixon 1973-79]



- **Compact body:** $\ell \propto \mu$
- **Replace the body smooth stress-energy tensor by a collection of multipole moments defined on a single worldline**

$$T^{\mu\nu}_{\rm skel}(x) = \sum_{l=0}^{+\infty} \frac{1}{l!} \int_\gamma \mathrm{d}\lambda \, I^{\mu\nu\alpha_1\dots\alpha_l}(z)\mathcal{D}^{(l)}_{\alpha_1\dots\alpha_l}\delta_4(x,z)$$

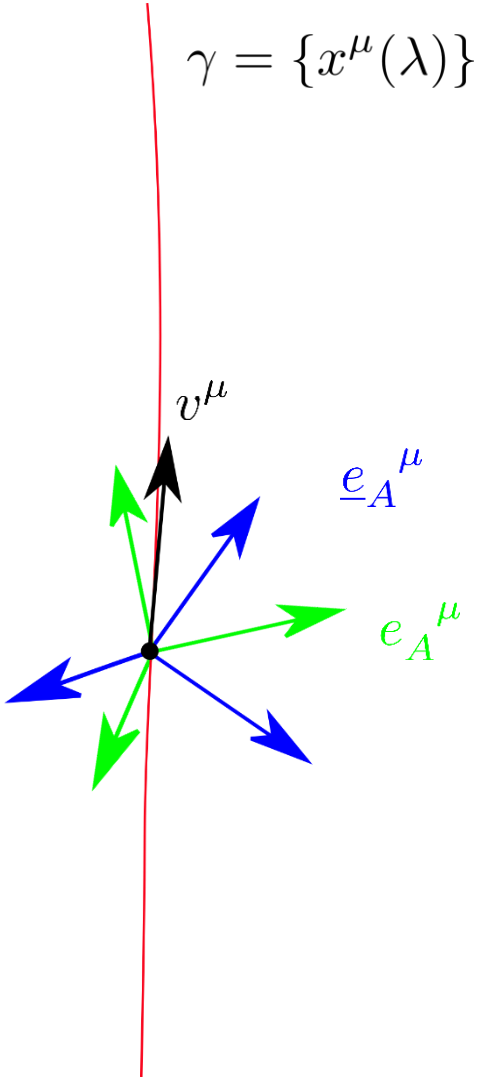
- **EOMs follow from** $\nabla_\mu T^{\mu\nu} = 0$
- **lth multipole scales as**

$$m\left(\frac{\ell}{r}\right)^l$$

l=0 monopole
l=1 dipole
l=2 quadrupole
…

Lagrangian formulation

[Hanson and Regge 1974, Bailey and Israel 1975]



- Body modelled as a worldline endowed with an orhtonormal tetrad $e_A{}^\mu$

$$S\left[x^\mu, e_A{}^\mu\right] = \int_\gamma L(v^\mu, \Omega^{\mu\nu}, g_{\mu\nu}(x^\alpha), R_{\mu\nu\rho\sigma}(x^\alpha), \nabla_\lambda R_{\mu\nu\rho\sigma}(x^\alpha), \ldots)\mathrm{d}\lambda$$

- Backgroud tetrad $\underline{e}_A{}^\mu$ for "reading" the orientation

$$\underline{e}_{\underline{A}}{}^\mu(z(\lambda)) = \Lambda^A{}_{\underline{A}}(\lambda) e_A{}^\mu(\lambda)$$

- Symmetry arguments allow to write

$$L = p_\mu v^\mu + \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu}$$

- Approximations <-> dependence of L in the Riemann tensor

$$\text{Dipole: } L = L(v^\mu, \Omega^{\mu\nu}, g_{\mu\nu}), \quad \text{Quadrupole: } L = L(v^\mu, \Omega^{\mu\nu}, g_{\mu\nu}, R_{\mu\nu\rho\sigma})$$

Mathisson-Papapetrou-Dixon equations

- **Both formulations lead to the MPD equations**

$$\frac{\mathrm{D}\,p^\mu}{\mathrm{d}\lambda} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} v^\nu S^{\alpha\beta} + \mathcal{F}^\mu, \qquad \mathcal{F}^\mu \triangleq -\frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^\mu R_{\alpha\beta\gamma\delta}$$

$$\frac{\mathrm{D}\,S^{\mu\nu}}{\mathrm{d}\lambda} = 2p^{[\mu} v^{\nu]} + \mathcal{L}^{\mu\nu}, \qquad \mathcal{L}^{\mu\nu} \triangleq \frac{4}{3} R^{[\mu}{}_{\alpha\beta\gamma} J^{\nu]\alpha\beta\gamma}$$

- **Interpretation of the linear momentum and the spin tensor**
  - **Skeletonization: multipole moments of the stress-energy tensor**

$$p^\mu = \int_{X^0=\mathrm{cst}} \mathrm{d}^3X \sqrt{-g}\, T^{\mu 0}, \qquad S^{\mu\nu} = \int_{X^0=\mathrm{cst}} \mathrm{d}^3X \sqrt{-g} \left(\delta X^\mu T^{\nu 0} - \delta X^\nu T^{\mu 0} \right)$$

  - **Lagrangian: momenta conjugated to 4-velocity and rotation coefficients**

$$p_\mu \triangleq \frac{\partial L}{\partial v^\mu}, \qquad S_{\mu\nu} \triangleq 2 \frac{\partial L}{\partial \Omega^{\mu\nu}}, \qquad J^{\mu\nu\rho\sigma} \triangleq -6 \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}$$

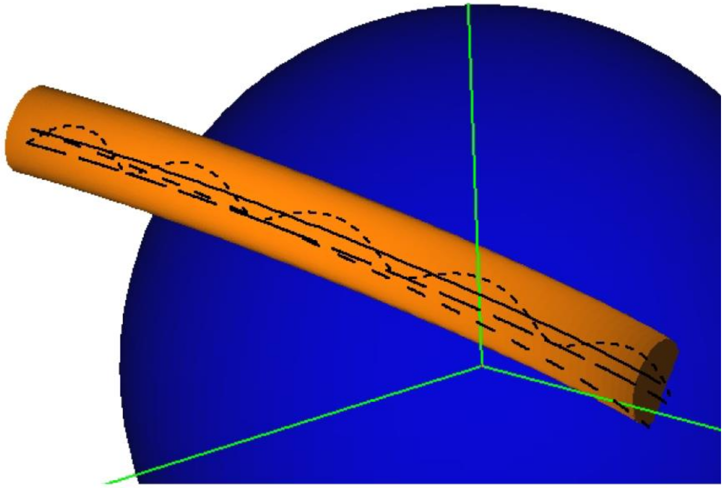
- **Spin magnitude** $\mathcal{S}^2 \triangleq \frac{1}{2} S_{\mu\nu} S^{\mu\nu}$ **exactly conserved**

Spin supplementary conditions

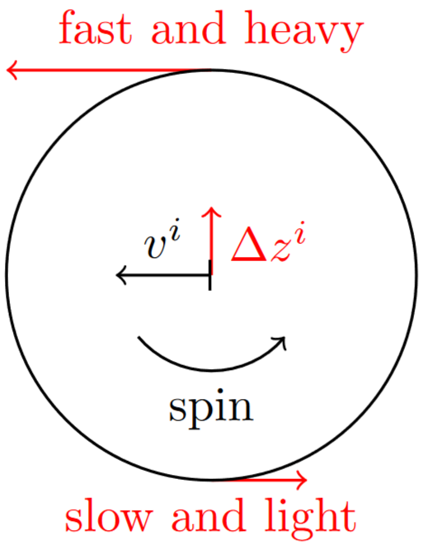
- 14 DOFs for 10 equations => system is not closed...
- Physical origin of the problem
  - Skelelonization: unspecified worldline
  - Lagrangian: only rotational DOFs of Lorentz tfo matter (boosts <=> gauge DOFs)
- Common choice: Tulczyjew-Dixon Spin Supplementary Condition (TD SSC)

$$p_\mu S^{\mu\nu} = 0 \qquad \Rightarrow \qquad S^{\mu\nu} = \frac{1}{2\mu}\epsilon^{\mu\nu\rho\sigma} p_\rho S_\sigma$$

- TD SSC + proper time: 4 additional constraints => closed system (relation between v and p)
- Specify the COM of the object (observer dependent)



[Kyrian and Semerák 2007]



[Steinhoff 2015]

Quadrupole approximation and astrophysics

- Quadrupole moment is not a dynamical variable => need for a prescription
- Discard tidal-type quadrupole deformations (BBH system)

Spin-induced quadrupole: most generic, well behaved quadrupole moment induced by the spin [Steinhoff 2014, Marsat 2015]

$$J^{\mu\nu\rho\sigma} = \kappa \frac{3p \cdot v}{(p^2)^2} p^{[\mu} S^{\nu]\lambda} S^{[\rho}{}_{\lambda} p^{\sigma]} = \mathcal{O}(\mathcal{S}^2)$$

- Coupling κ depends on the nature of the object (κ=1 for BH, 4<κ<8 for NS)

  => motion of test bodies not anymore universal @ quadrupole

- Perturbative expansion in S for astrophysically coherent systems, since

$$\frac{\mathcal{S}}{\mu M} \leq \frac{\mu^2}{\mu M} = \epsilon$$

III. Conserved quantities

Rüdiger's procedure for conserved quantities

1. Postulate an Ansatz

$$\mathcal{Q}^{(1)} \triangleq \sum_{p=1} Q^{[s,p]} \triangleq X_\mu p^\mu + W_{\mu\nu} S^{\mu\nu},$$

$$\mathcal{Q}^{(2)} \triangleq \sum_{p=2} Q^{[s,p]} \triangleq K_{\mu\nu} p^\mu p^\nu + L_{\mu\nu\rho} S^{\mu\nu} p^\rho + M_{\mu\nu\rho\sigma} S^{\mu\nu} S^{\rho\sigma}$$

2. Write down the conservation equation
3. Expand the conservation equation using the equations of motion
4. Express the conservation equation in terms of independent variables
5. Infer the independent constraints
6. Find solutions (and prove uniqueness)

Proc. R. Soc. Lond. A **375**, 185–193 (1981)
Printed in Great Britain

Conserved quantities of spinning test particles in general relativity. I

By R. Rüdiger
Institut für Astronomie und Astrophysik der Universität Würzburg, Am Hubland, D8700 Würzburg, F.R.G.

(*Communicated by R. Penrose, F.R.S. – Received 10 March 1980*)

This is the first of two papers devoted to conserved quantities of spinning test particles in general relativity. In this paper, a general scheme is described according to which these quantities can be investigated. It is shown that the general linear conserved quantity consists of a sum, the first term of which is the well known expression constructed from a Killing

Proc. R. Soc. Lond. A **385**, 229–239 (1983)
Printed in Great Britain

Conserved quantities of spinning test particles in general relativity. II†

By R. Rüdiger‡
Institut für Astronomie und Astrophysik der Universität Würzburg, Am Hubland, 8700 Würzburg, F.R.G.

(*Communicated by R. Penrose, F.R.S. – Received 11 August 1982*)

In this paper, which completes earlier work on conserved quantities of spinning test particles in general relativity (Rüdiger 1981*a*), quadratic conserved quantities are considered. It is shown that by a suitable change of variables the trivial conserved quantities, which result from a reducible Killing tensor, can essentially be separated from the non-trivial quantities. If the equations of motion are linearized in the spin, it is shown that non-trivial quantities of this type can be constructed for two classes of space-times including the Kerr geometry and the Friedman universe.

Example: geodesic motion

1. Assume the form of the invariant (polynomial)

$$C^{(n)}_{\mathbf{K}} \triangleq K_{\alpha_1 \dots \alpha_n} p^{\alpha_1} \dots p^{\alpha_n}$$

2. Plug it into the conservation equation

$$\dot{C}(p^\alpha) = 0 \quad \Leftrightarrow \quad v^\mu \nabla_\mu C(p^\alpha) = 0$$

3. Use EOMs to reduce it

$$p^\mu \nabla_\mu K_{\alpha_1 \dots \alpha_n} p^{\alpha_1} \dots p^{\alpha_n} = 0$$

4. Infer the constraints

$$\nabla_{(\mu} K_{\alpha_1 \dots \alpha_n)} = 0$$

5. Solve them !

"Symmetry implies conservation"
(textbook picture)

$\Updownarrow$

"Conservation requires symmetry"
(Rüdiger picture)

Status @ first order in the spin magnitude

- **Conservation understood in a perturbative sense:** $\dot{\mathcal{Q}} = \mathcal{O}(\mathcal{S}^2)$
- **Mass of the body μ conserved**
- **For any Killing vector** [Dixon 1979]

$$\mathcal{C}_\xi = \xi_\mu p^\mu + \frac{1}{2}\nabla_\mu \xi_\nu S^{\mu\nu}$$

- **In Ricci-flat spacetimes admitting a KY tensor** [Rüdiger 1981-83]

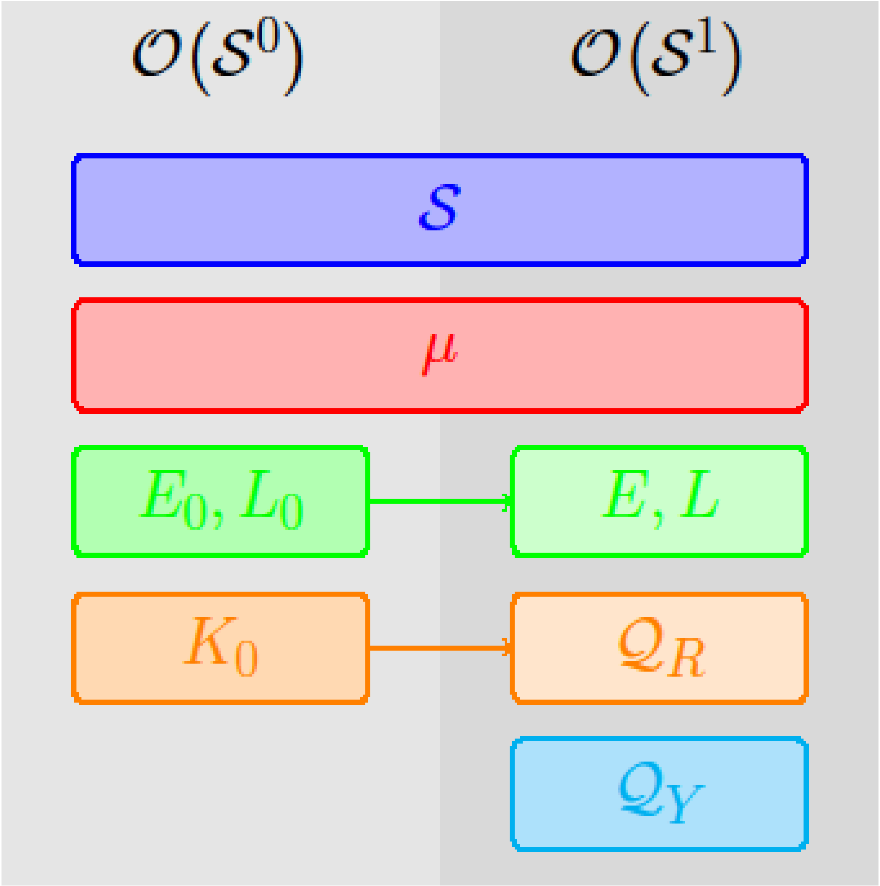
"linear" Rüdiger

$$\mathcal{Q}_Y = S^*_{\alpha\beta} Y^{\alpha\beta}$$

"quadratic" Rüdiger

$$\mathcal{Q}_R = Y_\mu{}^\lambda Y_{\nu\lambda} p^\mu p^\nu + 4\xi^\lambda \epsilon_{\lambda\mu\sigma[\rho} Y^\sigma{}_{\nu]} S^{\mu\nu} p^\rho$$

- **Unique in Kerr spacetime** [Compère and AD 2021]



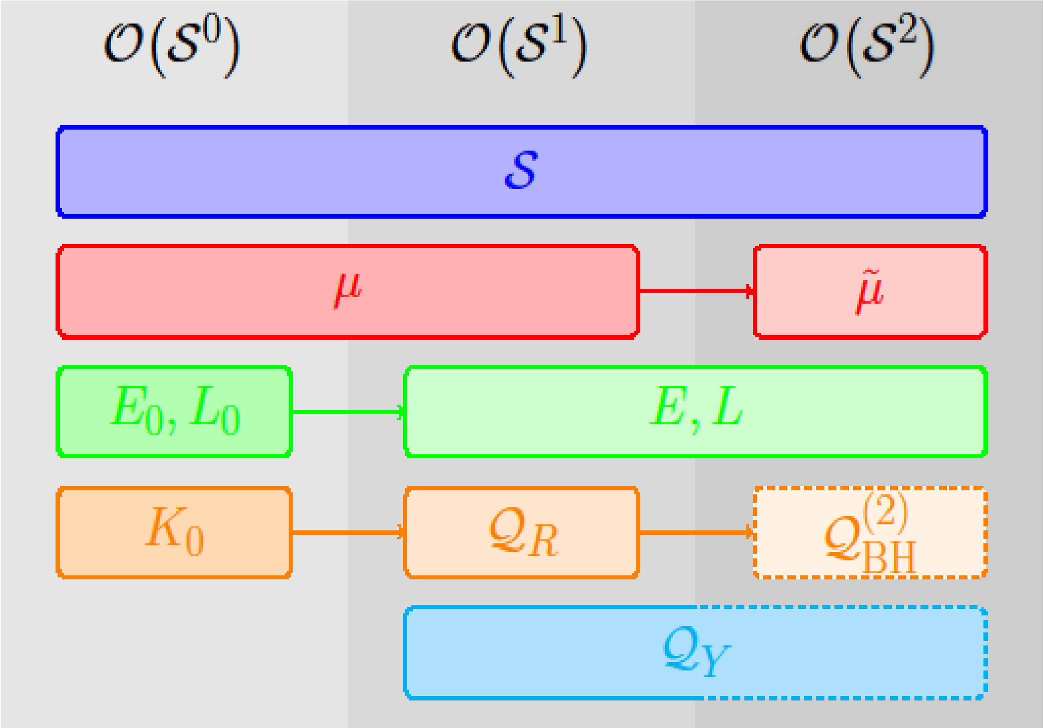
Status @ second order in the spin magnitude

[Compère, AD and Vines 2023]

- E, L, S still conserved
- Shifted mass-like quantity

$$\tilde{\mu} \triangleq \mu - \frac{1}{6} R_{\alpha\beta\gamma\delta} J^{\alpha\beta\gamma\delta}$$

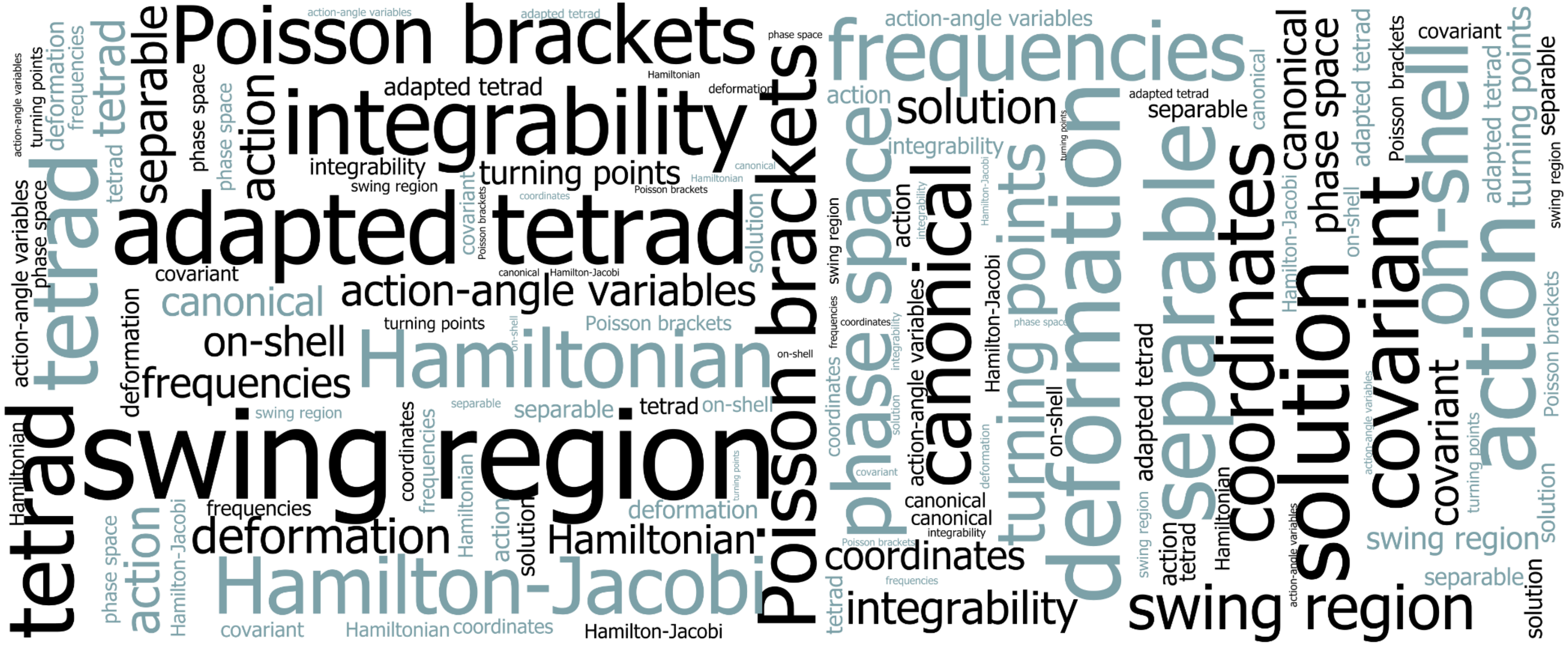
- "Hidden conserved quantities" require a new formalism
- For quadrupole BH coupling, linear Rüdiger still conserved + unique deformation of Carter constant
- No polynomial solution for other couplings



$$\mathcal{Q}^{(2)}_{\rm BH} = Y_\mu{}^\lambda Y_{\nu\lambda} p^\mu p^\nu + 4\xi^\lambda \epsilon_{\lambda\mu\sigma[\rho} Y^\sigma{}_{\nu]} S^{\mu\nu} p^\rho$$
$$+ \left[g_{\mu\rho}\left(\xi_\nu \xi_\sigma - \frac{1}{2} g_{\nu\sigma}\xi^2\right) - \frac{1}{2} Y_\mu{}^\lambda \left(Y_\rho{}^\kappa R_{\lambda\nu\kappa\sigma} + \frac{1}{2} Y_\lambda{}^\kappa R_{\kappa\nu\rho\sigma}\right)\right] S^{\mu\nu} S^{\rho\sigma},$$

Integrability ?

IV. Hamiltonian formulation



Symplectic structure and canonical coordinates

- **14D phase space, endowed with explicit PB structure**

$$\{x^\mu, x^\nu\} = 0$$
$$\{x^\mu, p_\nu\} = \delta^\mu_\nu$$
$$\{p_\mu, p_\nu\} = -\frac{1}{2} R_{\mu\nu\kappa\lambda} S^{\kappa\lambda}$$
$$\{S^{\mu\nu}, p_\kappa\} = 2\Gamma^{[\mu}_{\lambda\kappa} S^{\nu]\lambda}$$
$$\{S^{\mu\nu}, x^\kappa\} = 0$$
$$\{S^{\mu\nu}, S^{\kappa\lambda}\} = g^{\mu\kappa} S^{\nu\lambda} - g^{\mu\lambda} S^{\nu\kappa} + g^{\nu\lambda} S^{\mu\kappa} - g^{\nu\kappa} S^{\mu\lambda}$$

- **Canonical coordinates for the position sector** [Feldman et al. 1980]

$$P_\mu = p_\mu + \frac{1}{2} e^{\underline{A}}{}_{\alpha} e_{\underline{A}\beta;\mu} S^{\alpha\beta}$$

- **Spin sector is more involved**

[Witzany et al. 2019]

$$\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} \overset{V_\mu S^{\mu\nu}=0}{=} \mathcal{S}\left[\dot{\alpha} + \frac{\cos\vartheta - 1}{\sqrt{1-u^2}} \dot{\phi} + \left(\frac{1}{\sqrt{1-u^2}} - 1\right)\dot{\psi}\right] \Rightarrow \{x^\mu, P_\nu\} = \delta^\mu_\nu, \qquad \{\phi, A\} = \{\psi, B\} = 1$$

- **Equivalent, SSC free formulation by Ramond** [Ramond 2022]
- **TD SSC and mass-shell condition are first class constraints** [Steinhoff 2015]

Spinning particles: various Hamiltonians

- "Coordinate time" Hamiltonians: useful for PN applications [Barausse et al. 2009, Vines et al. 2016]

$$H = \beta^i p_i + \alpha\sqrt{\mu^2 + \gamma^{ij} p_i p_j} - E_{t\mu\nu} S^{\mu\nu}$$

- "Covariant" Hamiltonians more practical here
  - First order in the spin magnitude [Witzany et al. 2019]

$$H_{\text{lin}} = \frac{1}{2\mu} g^{\mu\nu} p_\mu \, p_\nu \approx -\frac{\mu}{2}$$

  - Second order in the spin magnitude, TD SSC with spin-induced quadrupole [AD and Ramond, to appear]

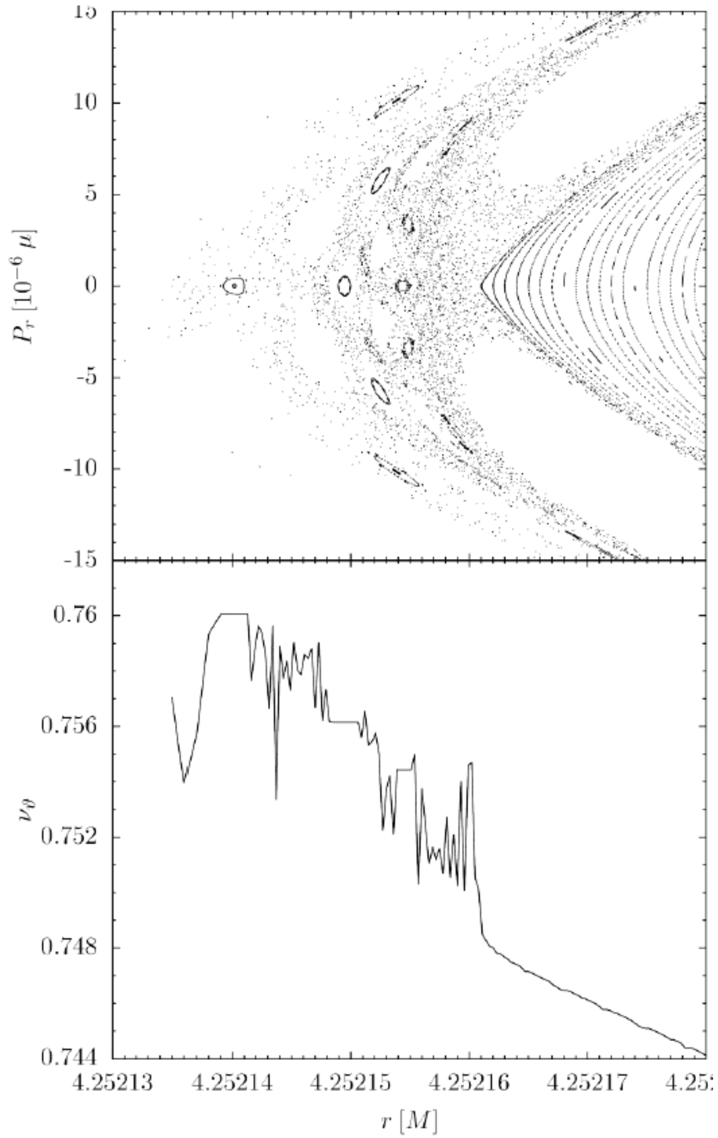
$$H_{\text{quad}} = \frac{1}{2\tilde{\mu}} \left[\left(\tilde{g}^{\mu\nu} + 2D^{\mu\nu} - \frac{\kappa}{2\tilde{\mu}^2} \left(\frac{g^{\rho\sigma} p_\rho p_\sigma}{\tilde{\mu}^2} + 1 \right) R^{\mu}{}_{\alpha\beta}{}^{\nu} \Theta^{\alpha\beta} \right) p_\mu p_\nu \right] \approx -\frac{\tilde{\mu}}{2}$$

Integrability ?

- Schwarzschild integrable at first order [Ramond 2022]
- Numerical studies in Schwarzschild [Zelenka et al. 2019] suggest that chaos arise in Schwarzschild at second order in the spin magnitude
- All feature of non-integrable perturbations visible (*c.f.* KAM theorem)
- Kerr non-integrable at linear order [Compère and AD 2021]

	Schwarzschild	Kerr
Linear	integrable	non integrable
Quadratic	non integrable	non integrable

	$\mathcal{S}$	E	ℓ	$\mathcal{Q}_Y$	$\mathcal{Q}_R$
μ^2	0	0	0	$\mathcal{O}(\mathcal{S}^2)$	$\mathcal{O}(\mathcal{S}^2)$
$\mathcal{S}$		0	0	0	0
E			0	0	$\mathcal{O}(\mathcal{S}^2)$
ℓ				0	$\mathcal{O}(\mathcal{S}^2)$
$\mathcal{Q}_Y$					$\mathcal{O}(\mathcal{S})$



[Zelenka et al. 2019]

Hamilton-Jacobi equation (I)

[Witzany 2019]

- Carter constant related to separability of HJ equation. What is the situation for MPD equations ?

$$1 + g^{\mu\nu} W_{,\mu} W_{,\nu} - W^{,\mu} \underline{e}_{\underline{A}\lambda;\mu} \underline{e}_{\underline{B}}{}^{\lambda} s^{\underline{A}\underline{B}} = \mathcal{O}(\mathcal{S}^2)$$

- Problem as a perturbation above geodesic motion: geodesic adapted tetrad

$$\underline{e}_{\underline{1}\mu} \underline{e}_{\underline{2}}{}^{\mu}{}_{;\nu} \underline{e}_{\underline{0}}{}^{\nu} = -\underline{e}_{\underline{2}\mu} \underline{e}_{\underline{1}}{}^{\mu}{}_{;\nu} \underline{e}_{\underline{0}}{}^{\nu} = \frac{\sqrt{K_c}}{\Sigma} \left(\frac{P(E_c, L_c, K_c)}{r^2 + K_c} + a \frac{L_c - aE_c(1 - x^2)}{K_c - a^2 x^2} \right)$$

- HJ equation is separable in the "swing region" $|r - r_t| \gg \mathcal{S}, \qquad r|x - x_t| \gg \mathcal{S}$

$$W^{(1\mathrm{sw})} = -E_{\mathrm{so}} t + L_{\mathrm{so}} \varphi + \left(s_{\parallel} - s \right) \phi + w_{1r}(r) + w_{1x}(x)$$

- Separation constant is Rüdiger quadratic invariant, and

$$E_{\mathrm{so}} = E, \quad L_{\mathrm{so}} = L, \quad s_{\parallel} = -\frac{1}{2} \frac{Q_Y}{\sqrt{K_0}} = \frac{S_\alpha \ell^\alpha}{\sqrt{\ell_\alpha \ell^\alpha}} \qquad K_0 = \ell_\alpha \ell^\alpha, \quad \ell_\alpha \triangleq Y_{\alpha\lambda} p^\lambda$$

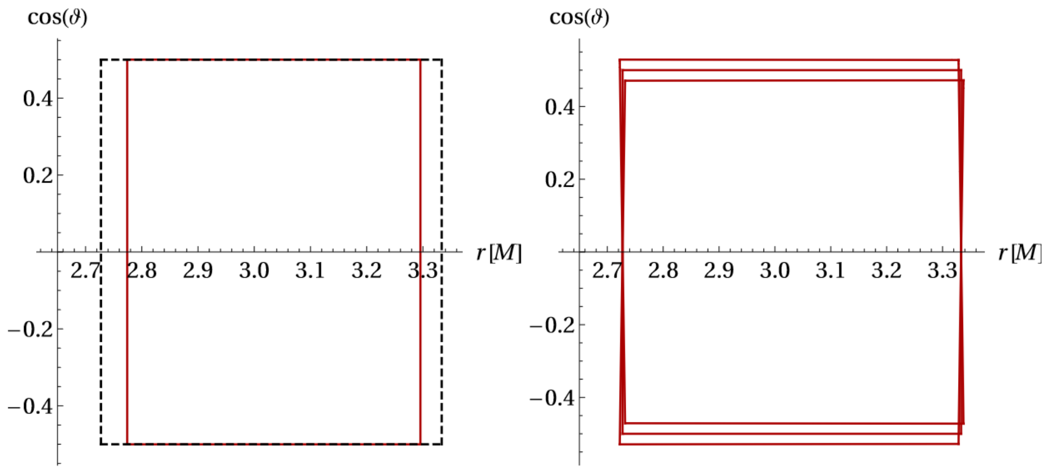
Hamilton-Jacobi equation (II)

[Witzany 2019]

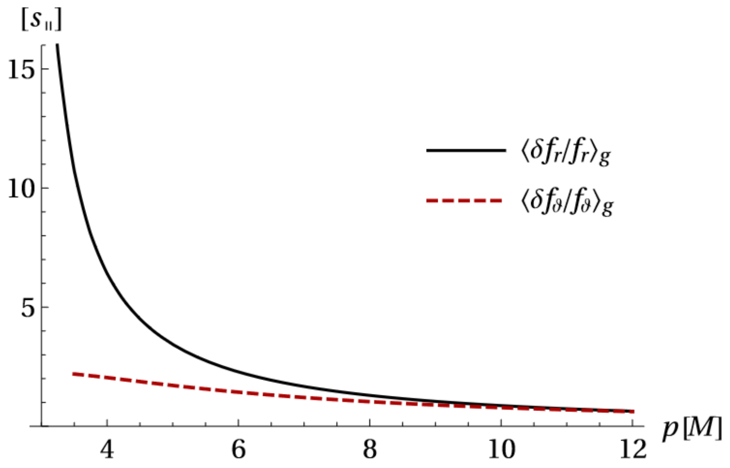
- Near the turning points, HJ equation is no more separable, but solvable as

$$W^{(1)} = -E_{\mathrm{so}}t + L_{\mathrm{so}}\varphi + \left(s_{\parallel} - s\right)\phi + \sum_{y=r,x} \int \left(\frac{1}{2} e_{\underline{A}\mu;y} e_{\underline{B}}{}^{\mu} \tilde{s}^{\underline{A}\,\underline{B}} \pm_y \sqrt{\left(w'_{1y}\right)^2 - e_{\underline{A}\mu;y} e_{\underline{B}}{}^{\mu} e_{0y} \tilde{s}^{\underline{A}\,\underline{B}}} \right) \mathrm{d}y$$

- Allows to provide corrections to the (radial and polar) turning points of the motion and to the fundamental frequencies of action-angle variables



[Witzany 2019]

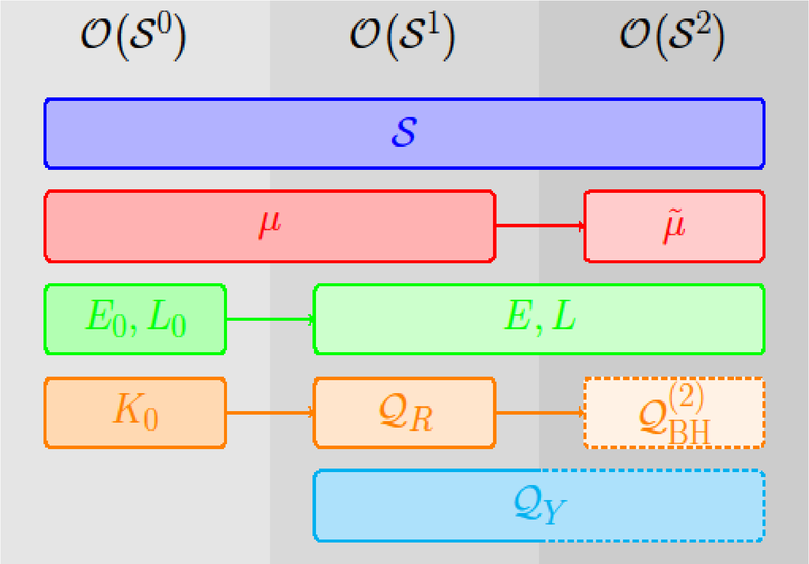


[Witzany 2019]

Conclusion and outlooks

	Schwarzschild	Kerr
Linear	integrable	non integrable
Quadratic	non integrable	non integrable

=> more analytical investigations ?



Conjecture. Hamilton-Jacobi equation for MPD equations (TD SSC + spin induced quadrupole with κ=1 BH coupling) is separable in the swing region at second order in the spin magnitude. The separated solution has the same form as at linear order, but the separation constant is the new quadratic invariant. [AD, to appear]

=> Compute the shifts in turning points and frequencies @ second order

Kerr is fun !!

Thanks for your attention !

Questions ?